The modularity of Calabi–Yau varieties
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1. Introduction.

A smooth projective variety X of dimension d is called a *Calabi–Yau* variety if
(1) $H^i(X, \mathcal{O}_X) = 0$ for every $i, 0 \leq i \leq d$, and
(2) X has the trivial canonical bundle.

We introduce the Hodge numbers $h^{i,j}(X) := \mathrm{dim H^j(X, \Omega^i_X)}$. Then the condition (1) is that $h^{i,0}(X) = 0$ for every $i, 0 < i < d$, and the condition (2) is that the geometric genus $p_g(X) := h^{d,0}(X) = 1$. The dimension one Calabi–Yau varieties are nothing but elliptic curves, and the dimension two Calabi–Yau varieties are $K3$ surfaces. The dimension three ones are Calabi–Yau threefolds. Note that Calabi–Yau threefolds are Kähler manifolds, so that $h^{1,1} > 0$. The Euler characteristic of a Calabi–Yau threefold is given by $\chi = 2(h^{1,1} - h^{2,1})$. Conjecturally, $|\chi|$ should be bounded, and the presently known bound is 960.

The (naive) mirror symmetry conjecture for Calabi–Yau threefolds asserts that given a Calabi–Yau threefold X, there is its mirror partner X^*, which is also a Calabi–Yau threefold, such that $h^{1,1}(X^*) = h^{2,1}(X)$, $h^{2,1}(X^*) = h^{1,1}(X)$ and $\chi(X^*) = -\chi(X)$.

In this lecture, we consider Calabi–Yau varieties as arithmetic objects. Assume that Calabi–Yau varieties are defined over $\mathbf{Q}$, or more generally, over a number field. The main theme of this talk is to discuss the modularity of Calabi–Yau varieties over $\mathbf{Q}$ (or a number field) in dimensions $1, 2$ and 3.

1. The modularity conjecture for Calabi–Yau varieties.

For dimension 1, the modularity conjecture has been established for all elliptic curves defined over $\mathbf{Q}$ by the celebrated efforts of Wiles and his former students.

Theorem (Wiles, et al.): *Every elliptic curve E over $\mathbf{Q}$ is modular. More precisely, the L-series of E is globally determined by a cusp form of weight 2 on $\Gamma_0(N)$ where N is the conductor of E.*

For dimension 2, the modularity conjecture has been proved for singular $K3$ surfaces. (Here "singular" means that the Néron–Severi group of a $K3$ surface has the maximal possible Picard number, namely, $20 = h^{1,1}(X)$.)

Theorem (Shioda and Inose): *Every singular $K3$ surface X is modular. More precisely, X has a model defined over a number field K and its L-series of X is given, up to a finitely many Euler factors, by $L(X, s) = \zeta_K(s-1)^{20} L(f, s)$ where f is a cusp form of weight 3 (possibly twisted by some character) on some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ (e.g., $\Gamma_1(N)$ or $\Gamma_0(N)$).*

For dimension 3, there is a special class of Calabi–Yau threefolds, *rigid* Calabi–Yau threefolds, for which the mirror symmetry conjecture fails (as a mirror partner does not have $h^{1,1}(X^*) > 0$).

The modularity conjecture for rigid Calabi–Yau threefolds over $\mathbf{Q}$: *Every rigid Calabi–Yau threefold over $\mathbf{Q}$ is modular.*

This conjecture may be regarded as a concrete realization of the conjecture of Fontaine and Mazur, which claims that every odd irreducible 2-dimensional Galois representation arising from geometry is modular.

Now we will formulate the modularity conjecture more precisely. Assume that a rigid Calabi–Yau threefold X has a suitable "integral" model. A rational prime p is called a *good* prime if X mod p is smooth and defines a rigid Calabi–Yau threefold over $\bar{\mathbf{F}}_p$. For a good prime, let $\mathrm{Frob_p}$ denote the Frobenius morphism, and let it act on $H^3_{\mathrm{et}}(X, \mathbf{Q}_\ell)$. Define $P_{p,3}(T) := \det(1 - \mathrm{Frob_p\, T} \,|\, \mathrm{H^3(X}, \mathbf{Q}_\ell))$. Then $P_{p,3}(T) = 1 - t_3(p)T + p^3T^2 \in \mathbf{Z}[\mathbf{T}]$ with $t_3(p) \in \mathbf{Z}$, $|t_3(p)| \leq 2p^{3/2}$. Furthermore, by the Lefschetz fixed point formula, $t_3(p)$ can be expressed in terms of the number of rational points on X over $\mathbf{F}_p$. Indeed, $t_3(p) =$

$1+p^3+(1+p)ph^{1,1}-\#X(\mathbf{F}_p)$ for all but finitely many p. The L-series of X is then defined by

$$L(X,s):=L(H^3_{\rm et}(X,\mathbf{Q}_\ell),s)=(*)\prod_{p:\text{good}}\frac{1}{1-t_3(p)p^{-s}+p^{3-2s}}$$

where $(*)$ is the factor corresponding to bad primes.

The Modularity Conjecture: *There is a cusp form f of weight 4 on some $\Gamma_0(N)$ where N is divisible only by bad primes, such that $L(X,s)=L(f,s)$.*

If X is not a rigid Calabi–Yau threefold over $\mathbf{Q}$, the modularity conjecture can still be formulated for a rank 2 motive in $H^3_{\rm et}(X,\mathbf{Q}_\ell)$.

2. Gathering evidence to the modularity conjecture.

Now we give evidence to the modularity conjecture for rigid Calabi–Yau threefolds. The construction of rigid Calabi–Yau threefolds is rather difficult, and we have at moment about fifteen rigid Calabi–Yau threefolds. Here are some of those examples.

The Schoen quintic. Let $Y: X_0^5+X_1^5+X_2^5+X_3^5+X_4^5-5X_0X_1X_2X_3X_4=0\subset\mathbf{P}^4$. Then Y has 125 nodes. Resolving singularities, one obtains a Calabi–Yau threefold X with $h^{3,0}=1, h^{2,1}=0, h^{1,1}=25$ and $\chi=50$.

Theorem (Schoen, Bloch, Serre, Faltings): *There is a cusp form f of weight 4 on $\Gamma_0(5^2)$ such that $L(X,s)=L(f,s)$. Furthermore, $f(q)$ can be expressed in terms of η-functions.*

The Hirzebrich quintic. Consider a regular pentagon in $\mathbf{R}^2$ with vertices $(u,0),(-\frac{1}{2},\pm\frac{u\sqrt{2-u}}{2})$, $(\frac{1-u}{2},\pm\frac{\sqrt{2-u}}{2})$ with $u=\frac{1+\sqrt{5}}{2}$. Taking the product of five lines defining the pentagon, we obtain the following affine equation $F(x,y)=(x+\frac{1}{2})(y^2-y^2(2x^2-2x+1)+\frac{1}{5}(x^2+x-1)^2)$. Let Y be the affine threefold defined by $F(x,y)-F(u,w)=0$. Then Y has 126 nodes. Blowing up Y along 126 nodes, we obtain a smooth Calabi- -Yau threefold X with $h^{3,0}=1$, $h^{2,1}=0$, $h^{1,1}=152$ and $\chi=152$. (Incidentally, this rigid Calabi–Yau threefold has the largest Euler characteristic among all rigid Calabi- -Yau threefolds known today.) From the defining equation, it is plain that 2 and 5 are bad primes.

Theorem (van Geemen and Werner): *There is a cups form f of weight 4 on $\Gamma_0(25^2)$ such that $L(X,s)=L(f,s)$. Here $f(q)$ is not expressible in terms of η- functions.*

The Verrill rigid Calabi–Yau threefold. Consider the root lattice A_3. All roots of A_3 are given by $\{\mathbf{e}_i-\mathbf{e}_j\,|\,0\le i,j\le 4,\ i\ne j\}$ where $\{\mathbf{e}_i\,|\,i=1,2,3,4\}$ is the standard basis for $\mathbf{R}^4$. To each root $\mathbf{e}_i-\mathbf{e}_j$, assign the monomial $X_iX_j^{-1}$. Summing over all the roots, and equating it with $\lambda\in\mathbf{P}^1$, one get a toric variety: $(X_1+X_2+X_3+X_4)(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1})=\lambda+4\subset\mathbf{P}^3\times\mathbf{P}^1$. This is not a Calabi–Yau threefold yet. To produce a Calabi–Yau threefold, we take the double cover of the base by putting $\lambda=\frac{(t-1)^2}{t}$. Then resolving singularities, one obtains a Calabi–Yau threefold X with $h^{3,0}=1$, $h^{2,1}=0$, $h^{1,1}=50$ and $\chi=100$.

Theorem (Verrill, M-H. Saito and Yui): *There is a cusp form f of weight 4 on $\Gamma_0(6)$ such that $L(X,s)=L(f,s)$. Here f is expressed in terms of η- functions, $f(q)=\eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2$.*

The complete intersection $\mathbf{P}^7(2,2,2,2)$. Let Y be the image of the complete intersection defined by the four equations:

$$Y_0^2=X_0^2+X_1^2+X_2^2+X_3^2$$
$$Y_1^2=X_0^2-X_1^2+X_2^2-X_3^2$$
$$Y_2^2=X_0^2+X_1^2-X_2^2-X_3^2$$
$$Y_3^2=X_0^2-X_1^2-X_2^2+X_3^2$$

Let X be a smooth resolution. Then X is a rigid Calabi- -Yau threefold with $H^{1,1}=128$.

Theorem (Nygaard and van Geemen): *There is a csup form f of weight 4 on $\Gamma_0(8)$ such that $L(X,s)=L(f,s)$. Here f is expressed in terms of η-functions, $f(q)=[\eta(q^2)\eta(q^4)]^4$.*

3. The methods for establishing the modularity conjecture.

We now discuss methods of establishing the modularity. There are at least four possible approaches in this regard. They are listed as follows.

(1) The Serre–Faltings criteria to establish equivalence of two 2-dimensional residual Galois representations. In our setting, this boils down to establish the equality: $t_3(p) =$ the p-th Fourier coefficient of $f(q)$ for all p (Serre), for finitely many p (Faltings). Also one needs the Chebotarev density theorem to conclude the equivanence of two 2-dimensional Galois representations involved.

This is the standard approach, and it is used to establish the modularity for the Schoen quintic, the Hirzebrich quintic, and also for the Verrill quintic (by H. Verrill).

(2) Wiles' approach. For $d = 1$, Wiles established equivalence of two 2-dimensional mod ℓ residual Galois representations arising from an elliptic curve and cusp form of weight 2 for $\ell = 3$. For rigid Calabi–Yau threefolds, apply Wiles' method for, say $\ell = 7$. Recently, the modularity of odd irreducible mod 7 2-dimensional Galois representations has been established.

(3) Geometric structures of rigid Calabi–Yau threefolds (toward classification of rigid Calabi–Yau threefolds). This is motivated by the classification theorem for singular $K3$ surfaces (due to Shioda and Inose) that every singular $K3$ surface is either a Kummer surface of product of two isogenous elliptic curves with CM, or a double cover of a Kummer surface of former type.

Lemma (Schoen): *Let Y be a relatively minimal rational elliptic surface with section. Assume that Y has exactly four singular fibers of type I_b, $b > 0$. Take the fiber product $Y \times_{\mathbf{P}^1} Y$ and then take its small resolution, X. Then X is a rigid Calabi–Yau threefold.*

We now apply Schoen's Lemma to elliptic modular surfaces. Let $\Gamma \subset PSL_2(\mathbf{Z})$ be a torsion-free arithmetic subgroup. Let C_Γ be the modular curve. Then there is a universal family of elliptic curves $\pi : S_\Gamma \to C_\Gamma$. The fibration π is called the *elliptic modular surface* associated to Γ. A natural question is: *Which Γ give rise to rigid Calabi–Yau threefolds via Schoen's construction?* The answer is given by the following Lemma.

Lemma (Beauville, Schoen, Sebbar): *There are only six subgroups, and they are:* $\Gamma(3)$, $\Gamma_0(4) \cap \Gamma(2)$, $\Gamma_1(5)$, $\Gamma_1(6) = \Gamma_0(6)$, $\Gamma_0(8) \cap \Gamma_1(4)$ *and* $\Gamma_0(9) \cap \Gamma_1(3)$.

Proposition *Let Γ be one of the six subgroups in the above Lemma. Then the rigid Calabi–Yau threefold X associated to Γ is modular, that is, $L(X, s) = L(f, s)$ where f is a cusp form of weight 4 on Γ.*

The proposition follows from the Shimura isomorphism, together with Deligne's work.

To establish the modularity for a given rigid Calabi–Yau threefold X, one tries to construct birational morphism defined over $\mathbf{Q}$ from X to one of the elliptic modular rigid Calabi–Yau threefolds. This method is applied by M-H. Saito and Yui to establish the modularity for the Verrill rigid Calabi–Yau threefold.

Theorem (M-H. Saito and Yui): *Let $Y : (X_1 + X_2 + X_3)(X_1X_2 + X_2X_3 + X_3X_1) = (s+1)X_1X_2X_3 \subset \mathbf{P}^2 \times \mathbf{P}^1$ be an elliptic surface. Then the minimal resolution of Y is isomorphic to the elliptic modular surface associated to $\Gamma_1(6) = \Gamma_0(6)$. Let X be the Verrill rigid Calabi–Yau threefold over $\mathbf{Q}$. Then X is birationally equivalent over $\mathbf{Q}$ to the minimal resolution of the fiber product $Y \times_{\mathbf{P}^1} Y$. Consequently, $L(X, s) = L(Y \times_{\mathbf{P}^1} Y, s) = L(f, s)$ where $f = \eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2$.*

(4) The intermediate Jacobians of rigid Calabi–Yau threefolds. For a Calabi–Yau threefold X, there are the Hodge decomposition and Hodge filtration:

$$H^3(X, \mathbf{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} = F^0 \supset F^1 \supset F^2 \supset F^3 \supset \{0\}$$

where $F^0 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}$, $F^2 = H^{3,0} \oplus H^{2,1}$, $F^2 = H^{3,0}$. The intermediate Jacobian $J^2(X)$ is defined by $J^2(X) = \frac{H^3(X,\mathbf{C})}{F^2H^3(X,\mathbf{C})+H^3(X,\mathbf{Z})}$. If X is rigid, $F^1 = F^2 = F^3 = H^{3,0}$ and $J^2(X) = H^{0,3}(X, \mathbf{C})/H^3(X, \mathbf{Z}) \simeq \mathbf{C}/\mathbf{Z}^2$ is a complex torus of dimension 1.

Question: *A rigid Calabi–Yau threefold X over $\mathbf{Q}$ is modular if the intermediate Jacobian $J^2(X)$ is defined over a number field and is modular.*

This approach is currently pursued jointly with Xavier Xarles (University Autonoma Barcelona).

4. Calabi–Yau varieties of CM type.

Let E be an elliptic curve Let End(E) denotes the endormorphism ring of E, and $\mathrm{End(E)} \otimes \mathbf{Q}$ the endormorphism algebra of E. It is known that $\mathrm{End(E)} \otimes \mathbf{Q}$ is isomorphic either to $\mathbf{Q}$, or an imaginary quadratic field over $\mathbf{Q}$. An elliptic curve E is said to have *complex multiplication* or simply *be of CM- type* if its endomorphism algebra $\mathrm{End(E)} \otimes \mathbf{Q}$ is an imaginary quadratic field over $\mathbf{Q}$. Of course, not all elliptic curves have CM. We try to generalize this notion to higher dimensional Calabi–Yau varieties.

Definition. A Calabi–Yau variety X of dimension ≥ 2 is said to *of CM type* if its Hodge group is commutative.

Now let X be a $K3$ surface. There is the direct decomposition of the second cohomology group

$$H^2(X, \mathbf{Q}) = NS(X) \oplus T(X)$$

where $NS(X)$ denotes the Néron–Severi group of X consisting of the classes of algebraic cycles on X, and $T(X)$ is the orthogonal complement of $NS(X)$ with respect to the cup product $< \quad , \quad >$. The Hodge group Hdg(X) is the smallest algebraic subgroup of $GL(T(X))$ containing the image of $U^1 := \{z \in \mathbf{C}^* \mid z = barz = 1\}$. A main result of Zarhin is that $T(X)$ is an irreducible Hodge module. So we may define the endormorphism algebra $\mathcal{E} := \mathrm{End_{Hdg}(T(X))} \otimes \mathbf{Q}$. Then $\mathcal{E}$ is a commutative field over $\mathbf{Q}$ with dimension dividing $\mathrm{rank_{\mathbf{Z}}T(X)}$. In fact, $\mathcal{E}$ is either a totally real algebraic number field, or a CM field over $\mathbf{Q}$.

Proposition *Let X be a singular $K3$ surface. Then the following assertions hold ture.*
(1) X is of CM type,
(2) $\mathcal{E}$ is an imaginary quadratic field over $\mathbf{Q}$.
(3) X is defined over a number field K, and the L–series of X is a product of 20 *copies of the Dedekind zeta-function of K and the L–series of a modular form of weight* 3 *on some congruence subgroup of* $\mathrm{SL}_2(\mathbf{Z})$.

Now we consider rigid Calabi–Yau threefolds. For rigid Calabi–Yau threefolds of CM type, we can establish the modularity conjecture. If X is a rigid Calabi–Yau threefold of CM type, then there is an elliptic curve over $\mathbf{Q}$ with CM such that the representations associated to the Calabi–Yau threefold do come from these representations, and this elliptic curve is (over $\mathbf{C}$) is the intermediate Jacobian.

Theorem (Xarles and Yui) *A rigid Calabi–Yau threefold of CM type is modular if and only if its intermediate Jacobian is of CM type, and is modular.*

Remarks (1) We should hasten to remark that not all rigid Calabi– Yau threefolds are of CM type, contrary to the dimension 2 case.

(2) The modularity conjecture for non-CM rigid Calabi–Yau threefolds is still open. Recently, the modularity of irreducible odd 2-dimensional Galois representations mod 7 has been established bu Manoharmayum. Use this result to establish the modularity of rigid Calabi–Yau threefolds over $\mathbf{Q}$ along the line of Wiles, and Taylor–Wiles.

(3) The Hodge group (or the Mumford–Tate group) of a Calabi–Yau threefold is not an esay object to compute. Therefore a compelling question is *How can one compute the Hodge group of a Calabi–Yau threefold?*

5. The modularity conjecture for Calabi–Yau varieties in general.

For $K3$ surfaces not necessarily of CM type, and for Calabi–Yau threefolds over number fields not necessarily rigid, the modularity conjecture ought to be formulated along the line of the Langland Program.

Suppose that Calabi–Yau varieties in question are defined over number fields $\mathbf{K}$. Let $\mathcal{G}$ denote the absolute Galois group. Then there is the Galois representation

$$\rho : \mathcal{G} \to GL(H^i_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell)$$

and there is the associated L–series $L(H^i_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell), s)$. The L-series of X is then defined by the d-th L-series $L(H^d_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell), s)$ where d is the dimension of X.

Theorem: *(a) A $K3$ surface of Picard number 19 is modular.*
(b) A Kummer surface is modular if and only if the associated abelian variety is modular.
(c) A $K3$ surface of Fermat type in a weighted projective space is modular.
(d) A $K3$ surface which is an orbifold of a surface of Fermat type in a weighted projective space is modular.

For Calabi–Yau threefolds, we have the following partial results.

Theorem: *(a) A Calabi–Yau threefold of Fermat type in a weighted projective space is modular.*
(b) A Calabi–Yau orbifold of a threefold of Fermat type in a weighted projective space is modular.
(c) A $K3$ fibered Calabi–Yau threefold is modular if and only if the $K3$ fiber is modular.

The next natural step in this research endevour might be to consider Calabi–Yau threefolds over $\mathbf{Q}$ with $h^{2,1} = 1$ or $H^{1,1} = 1$. Recently, Consani and Scholten considderd the affine singular Calabi–Yau threefold defined by the equation:

$$Y : H_t(x, y) - H_t(u, v) = 0$$

where $H_t(x, y) = F_{a,b}(x, y) = (x + a)(y^2 - x^2)(y^2 - b(x + 1)^2)$ with $a = t(t + 5)/(t^2 - 5); b = t^2/5$. Let X be a smooth resolution of Y. Then X is a Calabi–Yau threefold with $h^{2,1} = 1, h^{1,1} = 138$. The corresponding Galois representation is of dimension 4. They have determined the L-series of X over $\mathbf{Q}$. There is a Hilbert cusp form of weight $(2, 4)$ and level 5 such that $L(X, s) = L(f, s)$. Over $\mathbf{Q}(\sqrt{5})$, the L- series factors into the product of two L–series corresponding to the 2–dimensional Galois representations.

6. Concluding remarks and open problems.

6.1: A pair of $K3$ surfaces (X, X^*) is said to be a *mirror pair* in the sense of Dolgachev if

$$\mathrm{Pic(X)^\perp (= T(X)) = U = \oplus Pic(X^*)}.$$

In other words, a mirror pair of $K3$ surfaces (X, X^*) satisfies the identity:

$$\mathrm{rank_{\mathbf{Z}} Pic(X) + rank_{\mathbf{Z}} Pic(X^*) = 20}.$$

Since the Tate conjecture is true for any $K3$ surfaces in characteristic zero, this identity is further equivalent to the identity:

$$\mathrm{ord_{s=1} L(X, s) + ord_{s=1} L(X^*, s) = 20}.$$

The mirror symmetry phenomenon for $K3$ surfaces can be interpreted in terms of L–series.

6.2: A natural quesiton is : *Can one detect mirror symmetry for a mirror pair of Calabi–Yau threefolds?*

Consider the one-parameter family of quintic threefold defined by the hypersurface: $Y : X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - 5\psi X_0X_1X_2X_3X_4 = 0 \subset \mathbf{P}^4$ where ψ is a parameter. A smooth resolution of Y is a Calabi–Yau threefold X with $h^{3,0} = 1, h^{2,1} = 101, h^{1,1} = 1$ and $\chi = -200$. The mirror partner X^* exists.

Problem: *Interprete the mirror symmetry phenomenon in terms of zeta-functions and L-series of X and X^*.*

Candelas, de la Ossa and Villegas computed the zeta-functions of X and X^*, using the solutions of the GKZ hypergeometric system associated to the pair (X, X^*). The GKZ hypergeometric system contains the solutions to the Picard–Fuchs differential equation (the fundamental periods) but it also has extraneous solutions. These extraneous solutions do come into the description of the number $\#X(\mathbf{F}_p)$ of rational points on X. No explanations yet why this is so.

Stienstra also computed the zeta-functions of the mirror pair (X, X^*) of the quintic using formal deformations. His description of the zeta-functions involves the mirror map.