Semi-continuity of Hochschild Cohomology and Mesh Algebras without Outer Derivations

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Let R be a commutative noetherian ring with identity, and $A = \bigoplus_{i \geq 0} A_i$ a graded R-algebra satisfying the following:
(1) The R-module A_0 is free having a basis U that is a complete set of pairwise orthogonal idempotents of A.
(2) The R-module A_1 is finitely generated projective such that $A \cong T/I$ with T the tensor algebra of A_1 over A_0 and I a finitely generated homogeneous ideal.

An R-derivation $\delta : A \to A$ is of *degree zero* if $\delta(A_i) \subseteq A_i$ for all $i \geq 0$; and U-*normalzed* if $\delta(e) = 0$ for all $e \in U$. Let $\mathrm{Der}_R^U(A)_0$ denote the R-module of U- normalized derivations of degree zero and $\mathrm{Inn}_R^U(A)_0$ that of U-normalized inner derivations of degree zero. Then the first Hochschild cohomology group $\mathrm{HH}_R^1(A)$ of A over R contains as a submodule $\mathrm{HH}_R^1(A)_0 = \mathrm{Der}_R^U(A)_0/\mathrm{Inn}_R^U(A)_0$.

Using Grothendieck's semicontinuity theorem for homological functors, we have the following:

Theorem 1. *Let $A = \oplus_{i \geq 0} A_i \cong T/I$ be as above such that A_i is finitely generated projective for each degree i in which generators of I occur. Then*

$$p \mapsto \dim_{k(p)} \mathrm{HH}_{k(p)}^1(k(p) \otimes_R A)_0$$

is upper semicontinuous on $\mathrm{Spec}(R)$, *where $k(p)$ denotes the residue field of the localization R_p of R at p.*

Let Γ be a finite translation quiver without multiple arrows or loops. Say that Γ is *simply connected* if it contains no oriented cycle and its orbit graph is a tree. Denote by $R(\Gamma)$ the mesh algebra of Γ over R. As an application of the preceding result, we have the following:

Theorem 2. *If R is a noetherian domain, then the following are equivalent*:
(1) $\mathrm{HH}^1(R(\Gamma)) = 0$.
(2) *Γ is simply connected.*
(3) $\mathrm{HH}^1(R(\Delta)) = 0$ *for every connected convex translation subquiver Δ of Γ.*