Partition Theory and Banach Spaces, Lecture #5

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Definitions.

- $\mathrm{FIN}_k^{\pm} = \{s : \mathbb{N} \to \{-k, \dots, 0, \dots, k\} \mid \operatorname{supp} s \text{ finite},\ \max|s(i)| = k\}$
- $T : \mathrm{FIN}_k^{\pm} \to \mathrm{FIN}_{k-1}^{\pm}, \qquad T(s)(i) = \begin{cases} s(i) - 1, & s(i) > 0 \\ 0, & s(i) = 0 \\ s(i) + 1, & s(i) < 0. \end{cases}$
- $(A)_1 = \{s \mid \exists t \in A\ \|s - t\|_\infty \leq 1\}$ ("filtering by 1")
- An ultrafilter p on $\mathrm{FIN}_k^{\pm}$ is *cofinite* if $\{s \mid \operatorname{supp}(s) \cap n = \emptyset\} \in p$ for every $n \in \mathbb{N}$.
- An ultrafilter p on $\mathrm{FIN}_k^{\pm}$ is *subsymmetric* if for every $A \in p$, $-(A)_1 \in p$.
- $\gamma\,\mathrm{FIN}_k^{\pm}$ is the semigroup of all cofinite ultrafilters on $\mathrm{FIN}_k^{\pm}$. This is a closed subsemigroup of $\beta\,\mathrm{FIN}_k^{\pm}$.
- $S_k^{\pm}$ is the set of all cofinite subsymmetric ultrafilters on $\mathrm{FIN}_k^{\pm}$.
- $(px)\varphi(x)$ signifies that $\{x \mid \varphi(x)\} \in p$.
- Using this notation, we can rewrite the definition of $p + q$ for ultrafilters p and q as follows: $A \in p + q \Leftrightarrow (px)(qy)\, x + y \in A$.
- For $p \in \beta\,\mathrm{FIN}_k^{\pm}$, define $Tp \in \beta\,\mathrm{FIN}_{k-1}^{\pm}$ as follows:
$$Tp = \{T[A] \mid A \in p\}.$$
- For $p \in \beta\,\mathrm{FIN}_k^{\pm}$, define $-p \in \beta\,\mathrm{FIN}_k^{\pm}$ as follows:
$$A \in -p \Leftrightarrow \{-s \mid s \in A\} \in p.$$

Lemma 1. *$S_k^{\pm} \neq \emptyset$ for all $k \geq 1$.*

Proof. First, fix a sequence $(s_i)_{i=1}^{\infty}$ in $\mathrm{FIN}_k^{\pm}$ (for some fixed $k \geq 1$) such that $s_i < s_{i+1}$ (i.e., $\max \operatorname{supp} s_i < \min \operatorname{supp} s_{i+1}$) for all i. Define vectors

$$\begin{aligned} x_1 = T^{k-1}s_1 - T^{k-1}s_2 + T^{k-2}s_3 - T^{k-4}s_4 + \cdots \\ + Ts_{2k-1} - Ts_{2k} + s_{2k+1} - s_{2k+2} + Ts_{2k+3} - Ts_{2k+4} + \cdots \\ + T^{k-1}s_{4k-3} - T^{k-1}s_{4k-2}, \end{aligned}$$

$$\begin{aligned} x_2 = T^{k-1}s_2 - T^{k-1}s_3 + T^{k-2}s_4 - T^{k-4}s_5 + \cdots \\ + Ts_{2k} - Ts_{2k+1} + s_{2k+2} - s_{2k+3} + Ts_{2k+4} - Ts_{2k+5} + \cdots \\ + T^{k-1}s_{4k-2} - T^{k-1}s_{4k-1}. \end{aligned}$$

Notice that $\|x_1 + x_2\|_\infty \leq 1$, since

$$\begin{aligned} x_1 + x_2 = T^{k-1}s_1 + (T^{k-2}s_3 - T^{k-1}s_3) + (T^{k-3}s_5 - T^{k-2}s_5) + \cdots \\ + (s_{2k+1} - Ts_{2k+1}) + (Ts_{2k+3} - s_{2k+3}) + \cdots \\ + (T^{k-1}s_{4k-3} - T^{k-2}s_{4k-3}) - T^{k-1}s_{4k-1} \end{aligned}$$

and $\|T^{k-1}s_1\|_\infty \leq 1$, $\|T^{k-2}s_3 - T^{k-1}s_3\|_\infty \leq 1$, etc.; since the supports are disjoint and the norm is a sup-norm, the conclusion follows.

Now pick a cofinite $p \in \gamma\,\mathrm{FIN}_k^{\pm}$, and let

$$q = T^{k-1}p - T^{k-1}p + T^{k-2}p + T^{k-2}p + \cdots + p - p + \cdots + T^{k-1}p - T^{k-1}p.$$

Then $q \in \gamma\,\mathrm{FIN}_k^{\pm}$ by homework, so we only need to check that $A \in q \Rightarrow -(A)_1 \in q$.

If $A \in q$ then

$$\begin{aligned} &(ps_1)(ps_2)\cdots(ps_{4k-3})(ps_{4k-2})\, x_1 \in A \\ \Longrightarrow &(ps_1)(ps_2)\cdots(ps_{4k-3})(ps_{4k-2})(ps_{4k-1})\, x_1 \in A \\ \Longrightarrow &(ps_1)\cdots(ps_{4k-1})\, x_2 \in -(A)_1 \qquad (\|x_1 - (-x_2)\|_\infty \leq 1) \\ \Longrightarrow &(ps_2)\cdots(ps_{4k-1})\, x_2 \in -(A)_1 \\ \Longrightarrow &-(A)_1 \in q. \end{aligned}$$

□

Lemma 2. *$S_k^{\pm}$ is a closed subsemigroup of $\gamma\,\mathrm{FIN}_k^{\pm}$, and is closed under $-$.*

Proof. First, we check that it is closed. Pick some $p \notin S_k^{\pm}$; fix $A \in p$, $-(A)_1 \notin p$. Then $\{q \mid A \in q,\ \mathrm{FIN}_k^{\pm} \setminus -(A)_1 \in q\}$ is open and avoids $S_k^{\pm}$.

Next, we show that $S_k^{\pm}$ is a semigroup: Fix $p, q \in S_k^{\pm}$, $A \in p + q$. Recall the following definition: For $s \in \mathrm{FIN}_k^{\pm}$, $A \subseteq \mathrm{FIN}_k^{\pm}$, we defined $s^{-1}A = \{t \mid s + t \in A\}$. In a similar vein to $-(A)_1$, we now define $-(s^{-1})_1 = \{s' \mid \operatorname{supp} s' =$

$\operatorname{supp} s, \|s+s'\|_\infty \leq 1\}$. We now have

$$\begin{aligned}
&(ps)(qt)\; s+t \in A \\
\Longrightarrow &(ps)(qt)\; t \in s^{-1}A \\
\Longrightarrow &(ps)\; s^{-1}A \in q \\
\Longrightarrow &(ps)\; -(s^{-1}A)_1 \in q \\
\Longrightarrow &(ps)(qt)\; t \in -(s^{-1}A)_1 \\
\Longrightarrow &(ps)(qt)\; t \in s + -(A)_1 \\
\Longrightarrow &(ps)(qt)\; t \in -(s^{-1})_1 + -(A)_1 = -(s^{-1}A)_1
\end{aligned}$$

since W.L.O.G. $\operatorname{supp} r \cap \operatorname{supp} s = \emptyset\; \forall r \in A$

$$\Longrightarrow (ps)(qt)\; s+t \in -(A)_1.$$

$\square$

Lemma 3. $T : S_k^{\pm} \to S_{k-1}^{\pm}$.

Proof. Proved last time. $\square$

Lemma 4. *There are idempotents $p_k \in S_k^{\pm}$ such that $T(p_{k+1}) = p_k$ and $p_{k+1} \leq p_k$.*

Notice that if we could show $T(S_k^{\pm}) = S_{k-1}^{\pm}$, the proof would be the same as that of the analogous lemma for $\mathrm{FIN}_k^{\pm}$.

Proof. Let $R_{kj} = T^{k-j}(S_k^{\pm})$ (for $j < k$). This is a closed subsemigroup of $S_j^{\pm}$. Also, if $l \geq k$, then $R_{lj} \subseteq R_{kj}$. So $Q_j = \bigcap_{k>j} R_{kj}$ is a nonempty closed subsemigroup of $S_j^{\pm}$, and $T(Q_{j+1}) = Q_j$. Now go ahead and pick idempotents: Find $p_k \in Q_k$ using the earlier proof. $\square$

Theorem 5 (Gowers). *For every partition $\mathrm{FIN}_k^{\pm} = \bigcup_{i=1}^{l} Y_i$ there is a block basic sequence $A \subseteq \mathrm{FIN}_k^{\pm}$ such that $\langle A\rangle \subseteq (Y_i)_1$ for some $i \leq l$.*

Proof. Almost identical to the proof of the analogous theorem for FIN_k, but $A_0^k = Y_i \cap -(Y_i)_1$ for some i. Since p_k is subsymmetric, such a Y_i exists. $\square$

Theorem 6 (Gowers). *If $f : S_{C_0} \to \mathbb{R}$ is Lipschitz, $\epsilon > 0$, then there exists a block subspace $X \subseteq C_0$ for which $f[S_X]$ has diameter at most ϵ. (S_{C_0} is the unit sphere.)*

In order to prove this, we need to introduce another object, $\Delta_k^{\pm} \subseteq S_{C_0}$, defined for all $k > 1$. First, fix δ so that $(1+\delta)^{1-k} = \delta$. Now we let $\Delta_k^{\pm}$ be the space of all $p : \mathbb{N} \to \{0, \pm(1+\delta)^{1-k}, \pm(1+\delta)^{2-k}, \ldots, \pm(1+\delta)^{-1}, \pm 1\}$ such that $\operatorname{supp} p$ is finite and $\max_i p_i = 1$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(w) = \frac{\log|w|}{-\log(1+\delta)},\ \varphi(0) = +\infty.$$

Then $\varphi(\pm(1+\delta)^{-l}) = l$. Now define $\varphi : S_{C_0} \to \mathrm{FIN}_k^{\pm}$ by

$$\varphi(x)(n) = \operatorname{sign} x(n) \cdot \max\{0, k - \lfloor \varphi(x(n)) \rfloor\},$$

$\Psi = \varphi \restriction \Delta_k^{\pm}$. Note that $\varphi(-x) = \varphi(x)$ and that $\varphi(x+y) = \varphi(x) + \varphi(y)$ if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$.

Lemma 7. *For all $x \in \Delta_k^{\pm}$ and $0 \le \lambda \le 1$,*

$$\varphi(\lambda x) = T^j(\Psi(x)) \text{ for } j = \max\{k, \lfloor \varphi(\lambda) \rfloor\}.$$

Proof. If $\lambda = (1+\delta)^{-i}$ and $x(n) = (1+\delta)^{-l}$, then $\lambda \cdot x(n) = (1+\delta)^{-l-i}$. $\varphi(\lambda) = i$; then $\varphi((1+\delta)^{-l-i}) = k - l - i$ and $\Psi((1+\delta)^{l}) = k - l$. But $T^i(k-l) = \max\{k-l-i, 0\}$. $\square$

The following is an immediate corollary to Theorem 5 and Lemma 7, with k and δ as above.

Corollary 8 (Gowers). *For every finite partition of $\Delta_k^{\pm}$ there is an infinite-dimensional block subspace X of $C_0 \cap \Delta_k^{\pm}$ such that $S_X \cap \Delta_k^{\pm} \subseteq (p)_\delta$ for some piece p of the partition.*

Proof of Theorem 6. Let $\delta < \frac{\epsilon}{3}$, k satisfying $(1+\delta)^{1-k} = \delta$. Without loss of generality, $|f(x) - f(y)| \le \|x-y\|_\infty$ and $f : S_{C_0} \to [0,1]$. Define a partition of $\Delta_k^{\pm}$ into $\lceil \frac{3}{\epsilon} \rceil$ pieces, $\Delta_k^{\pm} = \bigcup_{i=0}^{\lceil \frac{3}{\epsilon} \rceil} Y_i$, where $x \in Y_i \Leftrightarrow \lceil \frac{3}{\epsilon} f(x) \rceil = i$. By Corollary 8, we have some i and an infinite block sequence $A \subseteq \Delta_k^{\pm}$ for which $\langle A \rangle \subseteq (Y_i)_\delta$. This means that $f[\langle A \rangle]$ has diameter at most $\frac{\epsilon}{3}$ (because f is Lipschitz). Since $\Delta_k^{\pm}$ is δ-dense in S_{C_0}, for all $x, y \in S_X$ (where X is the subspace of C_0 generated by A), $|f(x) - f(Y)| \le \epsilon$. $\square$