Partition Theory and Banach Spaces

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Introduction

The purpose of this section is to generalize the Finite Unions Theorem obtained in the previous lecture.

For a natural number k, define:

i) $\mathrm{FIN}_k = \{f \in \{0, \ldots, k\}^{\mathbb{N}} : |\mathrm{supp}(f)| < \aleph_0 \wedge f^{-1}\{k\} \neq \emptyset\}$.

ii) The tetris operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$:

$$\forall f \in \mathrm{FIN}_k, \forall n \in \mathbb{N}, \ \ T(f)(n) = \max(f(n) - 1, 0).$$

iii) Finally, if $Y = \langle y_n : n \in \mathbb{N}\rangle$ is a block sequence of elements of FIN_k (i.e. such that $\max(\mathrm{supp}(y_n)) < \min(\mathrm{supp}(y_{n+1}))$ for all $n \in \mathbb{N}$) let $\langle Y \rangle$ denote the set of all sums of the form:

$$\sum_{i=0}^{m} T^{m_i}(y_{n_i})$$

where $m \in \mathbb{N}$; $\forall i \in \{0, \ldots, m\}$, $m_i \in \{0, \ldots, k\}$, $n_i \in \mathbb{N}$, $n_0 < \ldots < n_m$, and at least one of the m_i's is 0.

Under these notations, the following holds:

Theorem (Gowers). *If FIN_k is partitioned into finitely many pieces $B_1, ..., B_l$, then one of the pieces B_i contains a block basic sequence Y such that $\langle Y \rangle \subset B_i$.*

A partition theorem for S^k

In what follows, $\tilde{T}$ denotes the map from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ given by:

$$\forall s \in \mathbb{N}^{\mathbb{N}}, \forall n \in \mathbb{N}, \ \ \tilde{T}(s)(n) = s(n+1).$$

Given a natural number $k \geqslant 1$, $\mathbb{N}^k$ is seen as $\{s \in \mathbb{N}^{\mathbb{N}} : supp(s) \subset \{0, \ldots, k-1\}\}$. In particular, if $s \in \mathbb{N}^k$, $\|s\|$ is the usual supremum norm of s.

Definition. $S^k = \{s \in \mathbb{N}^k : s(k-1) \neq 0\}$.

Definition. *Let $X = \langle x_n : n \in \mathbb{N}\rangle$ be a sequence of elements of S^k. Then define $\langle X \rangle$ to be the set of all sums of the form:*

$$\sum_{i=0}^{m} \tilde{T}^{m_i}(x_{n_i})$$

where $m \in \mathbb{N}$; $\forall i \in \{0, \ldots, m\}, m_i \in \{0, \ldots, k\}$, $n_i \in \mathbb{N}$, $n_0 < \ldots < n_m$, and at least one of the m_i's is 0.

The last part of this lecture will make clear the strong formal analogy between this definition and the one of $\langle Y\rangle$ given in introduction.

Equipped with pointwise addition and discrete topology, S^k naturally becomes a right-topological semigroup, for which we can we can consider the Čech-Stone compactification βS^k. Now, the advantage of seeing each $\mathbb{N}^k$ as a space of sequences is that if $i \leqslant j \in \mathbb{N}$, we can add any elements $s \in S^i$ and $t \in S^j$ ($s+t$ is then an element of $S^{max(i;j)}$), and this property transfers to the corresponding compactifications. More precisely:

Definition. *If $p \in \beta S^i$ and $q \in \beta S^j$:*

$$p+q = \{A \subset S^{\max(i,j)} : \{s \in S^i : -s+A \in q\} \in p\}$$

... where for $s \in S^i$, $-s+A = \{t \in S^j : s+t \in A\}$.

Note first of all that $p+q \in \beta S^{\max(i;j)}$.

Furthermore, it can be easily checked that this addition extends the usual addition on S^k to the corresponding compactification, and allows to consider the relation $p \leqslant q$ even if p and q are not idempotents in the same βS^k ; this just means $p+q = q+p = p$.

Similarly, $\tilde{T}$ can be extended to all the compactifications :

Definition. *For $k \in \mathbb{N}$, $p \in \beta S^k$:*

$$\tilde{T}(p) = \{A \subset S^{k-1} : \tilde{T}^{-1}A \in p\}$$

It is then easy to check that:

Proposition. *Let $k, j \in \mathbb{N}$, $p \in \beta S^k$, $q \in \beta S^j$. Then :*
i) $\tilde{T}(p) \in \beta S^{k-1}$.
ii) $\tilde{T}(p+q) = \tilde{T}(p) + \tilde{T}(q)$.

With these facts in mind, we can state the following lemmas:

Lemma 1. *There is a sequence $\langle p_k : k \in \mathbb{N} \setminus \{0\}\rangle$ such that for all k in $\mathbb{N}$:*
i) $p_k \in \beta S^k$ and $p_k + p_k = p_k$.
ii) $p_{k+1} \leqslant p_k$.
iii) $\tilde{T}(p_{k+1}) = p_k$.

Proof. We construct the p_k's inductively.

For p_1, pick any idempotent in βS^1.

Now, supposing that $p_1, p_2, \ldots, p_{k-1}$ are already defined, construct p_k as follows:

Let $X_k = \{q \in \beta S^k : \tilde{T}(q) = p_{k-1}\}$. We claim that X_k is not empty. To see this, notice that the family $(\tilde{T}^{-1}A)_{A \in p_{k-1}}$ of subsets of S^k has the finite intersection property and hence can be extended to an ultrafilter $q_k \in \beta S^k$. Now, by definition, $p_{k-1} \subset \tilde{T}(q_k)$ and since we are dealing with ultrafilters, $p_{k-1} = \tilde{T}(q_k)$. The claim is proved.

Moreover by continuity of $\tilde{T}$, X_k is closed. Thus, $X_k + p_{k-1} (= \{q + p_{k-1} : q \in X_k\})$ is nonempty and closed. We claim that it is also a subsemigroup of βS^k.

Indeed, let $q + p_{k-1}$, $r + p_{k-1}$ be in $X_k + p_{k-1}$.

Then $\tilde{T}(q + p_{k-1} + r) = p_{k-1} + p_{k-2} + p_{k-1} = p_{k-1}$.

Hence, $q + p_{k-1} + r \in X_k$ and so $q + p_{k-1} + r + p_{k-1} \in X_k + p_{k-1}$ which finishes the proof of the claim.

Now, pick an idempotent $q \in X_k + p_{k-1}$, and let p_k be $p_{k-1} + q$. It is $p_{k-1} + r + p_{k-1}$ for some $r \in X_k$, and so:

$$\begin{aligned} p_k + p_k &= (p_{k-1} + r + p_{k-1}) + (p_{k-1} + r + p_{k-1}) \\ &= p_{k-1} + r + p_{k-1} + r + p_{k-1} \\ &= p_{k-1} + q + q \\ &= p_{k-1} + q \\ &= p_k. \end{aligned}$$

In other words, p_k is an idempotent of βS^k. Since we clearly also have $p_k \leqslant p_{k-1}$ and $\tilde{T}(p_k) = p_{k-1}$, p_k fulfills the required conditions. $\square$

Lemma 2. *There is a sequence $\langle p_k : k \in \mathbb{N} \setminus \{0\}\rangle$ such that for all k in $\mathbb{N}$:*
i) $p_k \in \beta S^k$ and $p_k + p_k = p_k$.
ii) $\forall m \in \{0, \ldots, k\}, \tilde{T}^m(p_k) + p_k = p_k + \tilde{T}^m(p_k) = p_k$ (i.e. $\tilde{T}^m(p_k) \leqslant p_k$).

Proof. Apply lemma 1 and take the corresponding sequence $\langle p_k : n \in \mathbb{N} \setminus \{0\}\rangle$. Then for all $m \in \{0, \ldots, k\}, \tilde{T}^m(p_k) = p_{k-m} \geqslant p_k$. $\square$

Lemma 3. *Let $\langle x_n : n \in \mathbb{N}\rangle$ be a sequence of elements of S^k. Then for every $M \in \mathbb{N}$, there is a finite $B \subset \mathbb{N}$ such that every coordinate of $\sum_{i \in B} x_i$ is divisible by M.*

Proof. By pigeonhole principle, there is an infinite $C \subset \mathbb{N}$ and $t_0, \ldots, t_{k-1}$ such that: $\forall j \in \{0, \ldots, k-1\}$, $\forall i \in C$, $x_i(j) = t_j$. Take B to be any subset of C of cardinality M. $\square$

We are now ready to prove:

Theorem (Gowers). *If S^k is partitioned into finitely many pieces $A_1, ..., A_l$, then one of the pieces A_i contains a sequence $X = \langle x_n : n \in \mathbb{N}\rangle$ such that:*
i) $\langle X\rangle \subset A_i$.
ii) For every $n \in \mathbb{N}$, $2^{\|x_n\|}$ divides each coordinate of x_{n+1}.

Note that in what preceeds, ii) should really be seen as what it is: a technicality. However, it will play a crucial role in the proof of the main theorem, and this is the reason why it is included here.

Proof. For i) fix $p_1, \ldots, p_k$ as in lemma 1. Then for some $i \in \{0, \ldots, k\}$, $A_i \in p_k$.

The first step consists in constructing $A_i \supset A_0^k \supset A_1^k \supset \ldots$ a strictly decreasing sequence of elements of p_k, together with $\langle x_n : n \in \mathbb{N}\rangle$ such that if for each $h \in \{0, \ldots, k\}$ we let A_n^h be $\tilde{T}^{k-h}[A_n^k]$ (the direct image of A_n^k under $\tilde{T}^{k-h}$), then:
i) $x_n \in A_n^k$.
ii) $\forall i, j \in \{0, \ldots, k\}$, $\forall n \in \mathbb{N}$, $\{x \in S^k : \tilde{T}^i(x_n) + \tilde{T}^j(x) \in A_n^{\max(k-i,k-j)}\} \in p_k$.

Let $A_0^k = A_i$, and $i, j \in \{0, \ldots, k\}$. Since $A_0^k \in p_k$, we have $\tilde{T}^{\min(i,j)}[A_0^k] \in \tilde{T}^{\min(i,j)}(p_k)$.
But $\tilde{T}^{\min(i,j)}(p_k) = p_{k-\min(i,j)} = p_{k-i} + p_{k-j} = \tilde{T}^i(p_k) + \tilde{T}^j(p_k)$.
On the other hand, $\tilde{T}^{\min(i,j)}[A_0^k] = A_0^{\max(k-i,k-j)}$.
Consequently, $A_0^{\max(k-i,k-j)} \in \tilde{T}^i(p_k) + \tilde{T}^j(p_k)$, and:

$$\{y \in S^k : \{x \in S^k : \tilde{T}^i(y) + \tilde{T}^j(x) \in A_0^{\max(k-i,k-j)}\} \in p_k\} \in p_k$$

Call $B_{i,j}^0$ the set $\{y \in S^k : \{x \in S^k : \tilde{T}^i(y) + \tilde{T}^j(x) \in A_0^{\max(k-i,k-j)}\} \in p_k\}$.
From what preceeds,

$$A_0^k \cap \bigcap_{i,j \in \{0,\ldots,k\}} B_{i,j}^0 \in p_k$$

...so we can pick x_0 in this latter set, and i) and ii) hold.

In the general case, assume that $x_0, \ldots, x_n, A_0^k, \ldots, A_n^k$ have been defined such that i) and ii) hold at each stage. For $i, j \in \{0, \ldots, k\}$, let $C_{i,j}^n$ be the set:

$$C_{i,j}^n = \{x \in S^k : \tilde{T}^i(x_n) + \tilde{T}^j(x) \in A_n^{\max(k-i,k-j)}\}.$$

Every $C_{i,j}^n$ is in p_k by ii) and consequently:

$$\bigcap_{i,j \in \{0,\ldots,k\}} C_{i,j}^n \in p_k$$

Hence, if we set

$$A_{n+1}^k = A_n^k \cap \bigcap_{i,j \in \{0,\ldots,k\}} C_{i,j}^n$$

...then $A_{n+1}^k \in p_k$.

Now, as for A_0^k, $A_{n+1}^{\max(k-i,k-j)} \in \tilde{T}^i(p_k) + \tilde{T}^j(p_k)$.

Thus,

$$\{y \in S^k : \{x \in S^k : \tilde{T}^i(y) + \tilde{T}^j(x) \in A_{n+1}^{\max(k-i,k-j)}\} \in p_k\} \in p_k$$

Call $B_{i,j}^{n+1}$ the set $\{y \in S^k : \{x \in S^k : \tilde{T}^i(y) + \tilde{T}^j(x) \in A_{n+1}^{\max(k-i,k-j)}\} \in p_k\}$.

From what preceeds,

$$A_{n+1}^k \cap \bigcap_{i,j \in \{0,\ldots,k\}} B_{i,j}^{n+1} \in p_k$$

...so we can pick x_{n+1} in this latter set, and i) holds.

Furthermore, if we call $C_{i,j}^{n+1}$ the set $\{x \in S^k : \tilde{T}^i(x_{n+1}) + \tilde{T}^j(x) \in A_{n+1}^{\max(k-i,k-j)}\}$, then every $C_{i,j}^{n+1}$ is in p_k because $x_{n+1} \in B_{i,j}^{n+1}$ and so ii) holds, and the construction can be carried on.

The second step is to prove the following claim:

Claim. *If $h \in \mathbb{N}, n_0 < \ldots < n_h \in \mathbb{N}$ and $m_0, \ldots, m_h \in \{0, \ldots, k\}$, then:*

$$\sum_{i=0}^{h} T^{m_i}(x_{n_i}) \in A_{n_1}^{\max\{k-m_i : i \in \{1,\ldots,h\}\}}$$

...so in particular $\langle x_{n_i} : i \in \mathbb{N} \rangle \subset A_0^k \subset A_i$.

The proof goes by induction on h.

For $h = 0$, we have to show that given $n \in \mathbb{N}$ and $m \in \{0, \ldots, k\}$, $\tilde{T}^m(x_n) \in A_n^{k-m}$. But this means $x_n \in A_n^k$ ...which is true by construction.

Now assume that the claim is true for a fixed $h \in \mathbb{N}$. We want to prove that it is also true for $h + 1$. Fix $n_0 < \ldots < n_{h+1} \in \mathbb{N}$ and $m_0, \ldots, m_{h+1} \in \{0, \ldots, k\}$. By induction hypothesis:

$$\sum_{i=1}^{h+1} T^{m_i}(x_{n_i}) \in A_{n_1}^{\max\{k-m_i : i \in \{1,\ldots,h+1\}\}} = A_{n_1}^{k-m} = \tilde{T}^m[A_{n_1}^k]$$

...where $m = \min\{m_i : i \in \{1, \ldots, h+1\}\}$.

Consequently, there is $y \in A_{n_1}^k$ such that $\tilde{T}^m(y) = \sum_{i=1}^{h+1} T^{m_i}(a_{n_i})$. Now, by construction:

$$A_{n_1}^k \subset A_{n_0+1}^k \subset C_{m_0,m}^{n_0} = \{x \in S^k : \tilde{T}^{m_0}(x_{n_0}) + \tilde{T}^m(x) \in A_{n_0}^{\max(k-m_0,k-m)}\}.$$

Hence, $\tilde{T}^{m_0}(x_{n_0}) + \tilde{T}^m(y) \in A_{n_0}^{\max(k-m_0,k-m)} = A_{n_0}^{\max\{k-m_i : i \in \{0,\ldots,h+1\}\}}$.

In other words:

$$\sum_{i=0}^{h+1} T^{m_i}(x_{n_i}) \in A_{n_0}^{\max\{k-m_i : i \in \{0,\ldots,h+1\}\}}$$

...and this ends the proof for i).

For ii), we actually replace X which we just defined by $X' = \langle X_n' : n \in \mathbb{N} \rangle$ constructed inductively as follows:

Take $x_0' = x_0$.

Assuming that $x_0', \ldots, x_n'$ are defined, using lemma 3, find a finite $B_{n+1} \subset \mathbb{N}$ so that every coordinate of $\sum_{i \in B_{n+1}} x_i$ is divisible by $2^{\|x_n'\|}$, and set $x_{n+1}' = \sum_{i \in B_{n+1}} x_i$.

Then X' clearly satisfies ii). It also satisfies i) since $\langle X' \rangle \subset \langle X \rangle$. $\square$

Gowers's theorem for FIN_k

Definition. *For a finite $A \subset \mathbb{N}$, $\hat{A} = \sum_{i \in A} 2^i$.*

This defines a bijection from $[\mathbb{N}]^{<\aleph_0}$ to $\mathbb{N}$. Notice also that $\hat{A} = 0$ iff $A = \emptyset$, and that when $A, B \subset \mathbb{N}$ are disjoint, $\widehat{A \cup B} = \hat{A} + \hat{B}$.

Definition. *For $f \in \mathrm{FIN}_k$, $\phi(f) = (\widehat{f^{-1}\{1\}}, \ldots, \widehat{f^{-1}\{k\}}) \in \mathbb{N}^k$.*

Since $f^{-1}\{k\} \neq \emptyset$, this defines a map from FIN_k to S^k, which turns out to be bijective and to verify the following immediate properties:

Lemma 4. *Let $f, g \in \mathrm{FIN}_k$ and $m \in \mathbb{N}$.*
i) $\phi(T^m(f)) = \tilde{T}^m(\phi(f))$.
ii) If f and g have disjoint supports, then $\phi(f+g) = \phi(f) + \phi(g)$.

Lemma 5. *Let $j \in \mathbb{N}$, and $f \in \mathrm{FIN}_k$.*
i) If $\|\phi(f)\| < 2^j$, then $\max(\mathrm{supp}(\phi(f))) < j$.
ii) If 2^j divides every coordinate of $\phi(f)$, then $j \leqslant \min(\mathrm{supp}(\phi(f)))$.

We can now prove the theorem given in introduction:

Theorem. *If FIN_k is partitioned into finitely many pieces $B_1, ..., B_l$, then one of the pieces B_i contains a block basic sequence Y such that $\langle Y \rangle \subset B_i$.*

Proof. Suppose that FIN_k is partitioned into finitely many pieces $B_1, \ldots, B_l$. Via ϕ, this induces a partition $A_1, \ldots, A_l$ of S^k. Take any sequence $X = \langle x_n : n \in \mathbb{N} \rangle$ given by the partition theorem for S^k. X is the image of some sequence $Y = \langle y_n : n \in \mathbb{N} \rangle$ in FIN_k by ϕ, ie $\forall n \in \mathbb{N}$, $x_n = \phi(y_n)$. We claim that Y fulfills all the requirements.

Actually, thanks to lemma 4, it suffices to show that Y is a block sequence...But this follows from the properties of X:

If $n \in \mathbb{N}$, then $\|x_n\| < 2^{\|x_n\|}$ and $2^{\|x_n\|}$ divides each coordinate of x_{n+1}. By lemma 5, for every $n \in \mathbb{N}$, $\max(\mathrm{supp}(y_n)) < \|x_n\| \leqslant \min(\mathrm{supp}(y_{n+1}))$. Hence Y is block and this finishes the proof.

□