# Partition Theory and Banach Spaces, Lecture #3

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## Statement of Hindman's Theorem

So far, we have seen only one Pigeonhole Principle (Ramsey's Theorem), but there are others.

**Theorem (Hindman's Theorem).** *If $A \in [\mathbb{N}]^\omega$ and $f : \mathbb{N} \to k$, then there is a $B \in [\mathbb{N}]^\omega$ and $i < k$, so that each $m \in B$ is equal to a sum of some elements of $A$, and $f(\sum S) = i$ for every finite $S \subseteq B$.*

Note that this is quite different from Ramsey's Theorem.

It is possible to prove Hindman's Theorem using brute force, but we are going to develop some elegant machinery to prove the theorem.

## Idempotents in Compact Semigroups

**Definition.** *Suppose we have a semigroup $(S, *)$, and a topology on $S$, so that $\varrho_a : S \to S$, $\varrho_a(b) = b * a$ (the right translation by $a$) is continuous for every $a \in S$.*

*This is called a **right-topological semigroup**, and the semigroup operation * is said to be **right continuous**.*

(Note that we don't require multiplication to be continuous. We don't even require left translation to be continuous.)

Note that, alternatively, the theory can be developed by working with **left-topological semigroups**, defined in the obvious way. However, some authors revererse the notation, defining left- and right-topological semigroups with meanings opposite to the ones given here. Therefore, there are altogether 4 different ways of presenting this theory!

On page 31 of *Algebra in the Stone-Čech Compactification – Theory and Applications* (W. de Gruyter & Co., Berlin, 1998), N. Hindman and D. Strauss write:

> We should note that many authors use the term "left topological" for what we call "right topological" and vice versa. One may reasonably ask why someone would refer to an operation for which multiplication on the right is continuous as "left continuous". The people who do so ask why we refer to an operation which is continuous in the left variable as "right continuous".

**Example.** *Let $X$ be a Hausdorff topological space. Let $S = X^X$ with the product topology. Let the semigroup operation be composition of functions. Then $\varrho_f$ is continuous for every $f \in X^X$. Therefore, this is an example of a right-topological semigroup. Note that the left translation is not always continuous in this case.*

We'll be interested in right topological semigroups that are compact.

**Theorem (1: Auslander-Ellis).** *Every compact right-topological semigroup $S$ has an idempotent. (Recall that an idempotent is an element $x \in S$ such that $x^2 = x$.)*

*Proof.* Consider the family $\mathcal{F}$ of all compact subsemigroups of $S$. Since $\mathcal{F}$ is closed under intersections of chains, by Zorn's Lemma $\mathcal{F}$ has a minimal element $R$. Pick $s \in R$. $Rs = \{as : a \in R\}$ is a continuous image of $R$, so it is compact. $(as)(bs) = (asb)s \in Rs$ so $Rs$ is a semigroup. Since $s \in R$, $Rs \subseteq R$. By the minimality of $R$, $Rs = R$.

Let $P = \{x \in R \mid xs = s\} = \varrho_s^{-1}(\{s\})$. $P$ is closed. $P \subseteq R$ so $P$ is compact.

**Claim.** *$P$ is a semigroup.*

*Proof of Claim.* Pick $a \in P$, $b \in P$. $(ab)s = a(bs) = as = s$. So $ab \in P$. □

$P$ is a compact subsemigroup, $P \subseteq R$, so $P = R$. So $s \in P$ and $s^2 = s$. $s$ is the required idempotent. □

Assume throughout that $S$ is a compact right-topological semigroup.

**Definition (2).** *A **right ideal** in $S$ is a closed, nonempty $J \subseteq S$ such that $JS \subseteq J$.*
*A **left ideal** in $S$ is a closed, nonempty $J \subseteq S$ such that $SJ \subseteq J$.*
*A **two-sided ideal** in $S$ is $J$ that is both a left and right ideal.*

**Lemma (3).** *If $J$ is a minimal (under inclusion) left ideal, then so is $Jx$, for any $x \in S$. Moreover, $Jx = J$ if $x \in J$.*

**Definition (6).** *If $x$ and $y$ are idempotents in $S$, write $x \leq y \iff xy = yx = x$. (This is a partial order, but not a linear order.)*

**Lemma (7).** *If an idempotent $y$ belongs to an a minimal left-ideal $J$, then it is $\leq$-minimal.*

*Proof.* We need to check that $x \leq y$ implies $x = y$ (for $x^2 = x$). Assume $xy = yx = x$. Since $xy = x$ and $xy \in J$, we have $x \in J$. So, by Lemma 3, $Jx$ is a minimal left-ideal, and $Jx \subseteq J$. By the minimality of $J$, $Jx = J$. So $y \in Jx$. Pick $z \in J$ such that $y = zx$. $x = yx = (zx)x = zx^2 = zx = y$. □

**Lemma (9: Key Lemma).** *For every idempotent $y$ and every left ideal $J$, the left ideal $Jy$ contains an idempotent $x$ such that $x \leq y$.*

*Proof.* By Theorem 1 (applied to $Jy = S$), there is an idempotent $w \in Jy$. $SJy \subseteq Jy$. Pick $v \in J$ such that $w = vy$. Let $x = yw$. $yw \in SJy \subseteq Jy$ so $x \in Jy$. $x^2 = (yw)(yw) = (yvy)(yvy) = yvyvy = yw^2 = yw = x$. Check $x \leq y$: $xy = ywy = yvy^2 = yvy = yw = x$. $yx = y(yw) = yw = x$. □

**Theorem (10).** *An idempotent is minimal iff it belongs to some minimal left ideal.*

*Proof.* ($\Leftarrow$): Lemma 7.

($\Rightarrow$): Assume $y$ is a minimal idempotent. Pick any minimal left ideal $J$. Then $Jy$ is a minimal left ideal. We need to check that $y \in Jy$. By Lemma 9, there is an idempotent $x \in Jy$ such that $x \leq y$. Since $y$ is minimal, $x = y$. □

**Theorem (11).** *If $y$ is an idempotent and $J$ is a two-sided ideal, then there is an idempotent $x \in J$ such that $x \leq y$.*

*Proof.* Since $J$ is a left ideal and $Jy \subseteq J$, the theorem follows from the Key Lemma. □

**Lemma (12: not needed).** *If $J$ is a two-sided ideal, then it contains every minimal left ideal.*

## The Semigroup Operation on the Čech-Stone Compactification $\beta S$

Let $(S, *)$ be an infinite semigroup (think of $S = (N, +)$) with the discrete topology. Define a right-topological semigroup as follows: $\beta S$ is the Cech-Stone compactification of $S$; its elements are all ultrafilters on $S$. (Identify $a \in S$ with with the principal ultrafilter $\{A \subseteq S \mid a \in A\}$, so that $S \subseteq \beta S$.)

Recall the following

**Lemma.** *If $K$ is compact and $f : S \to K$ is continuous, then there is the unique continuous extension $\beta f$ of $f$, $\beta f : \beta S \to K$.*

*Proof.* The function $\beta f$ is constructed by

$$\beta f(P) = \lim_{a \to P} f(a).$$

The proof was given as an exercise in the Summer Course. In particular, it was Exercise 1 given in the lecture by Prof. Juris Steprans, on Tuesday, August 27, 2002. $\square$

Note that in our context, the assumption of continuity of $f$ is redundant since $S$ has the discrete topology.

Now, extend the semigroup operation * to $\beta S$ as follows:

Step 1: For $a \in S$, define $\lambda_a : S \to S$ by $\lambda_a(b) = a * b$. By the lemma, there is the unique continuous extension $\hat{\lambda}_a : \beta S \to \beta S$ of $\lambda_a$.

Step 2: For $a \in \beta S$, define the continuous $\varrho_a : S \to \beta S$ by $\varrho_a(b) = b * a = \hat{\lambda}_b(a)$. Extend $\varrho_a$ to the continuous $\hat{\varrho}_a : \beta S \to \beta S$.

Step 3: Let $a * b = \hat{\varrho}_b(a)$.

This extends the semigroup operation * to $\beta S$. The operation * on $\beta S$ is associative (check!) and right continuous, but not necessarily left continuous. So, $(\beta S, *)$ is a compact right topological semigroup.

As an alternative definition of *: For $p, q \in \beta S$, let

$$p * q = \lim_{a \to p} \left( \lim_{b \to q} a * b \right).$$

The proof of Hindman's Theorem will be given in the next lecture, by considering the Čech-Stone compactification of the naturals, $(\beta \mathbb{N}, +)$.