Quantum error correction for continuously detected errors

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Outline

- Two methods of controlling quantum systems: Quantum error correction vs. quantum feedback

- Combining quantum error correction with quantum feedback: controlling/protecting state by correcting for a specific error process

  - Two-qubit code + driving Hamiltonian for spontaneous emission
  - n-qubit code + driving Hamiltonian, protecting (n-1) logical qubits for arbitrary error channel

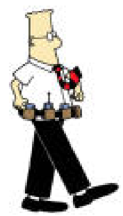
Dealing with hostile environments
or: Quantum error correction (QEC) and Quantum control (QC)

Goal:

- QC: protect quantum state
- QEC: protect unknown quantum state

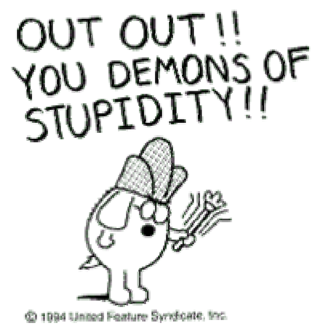
Environment

- QC: continuous measurements
- QEC: projective measurements



System

Observe



Feedback

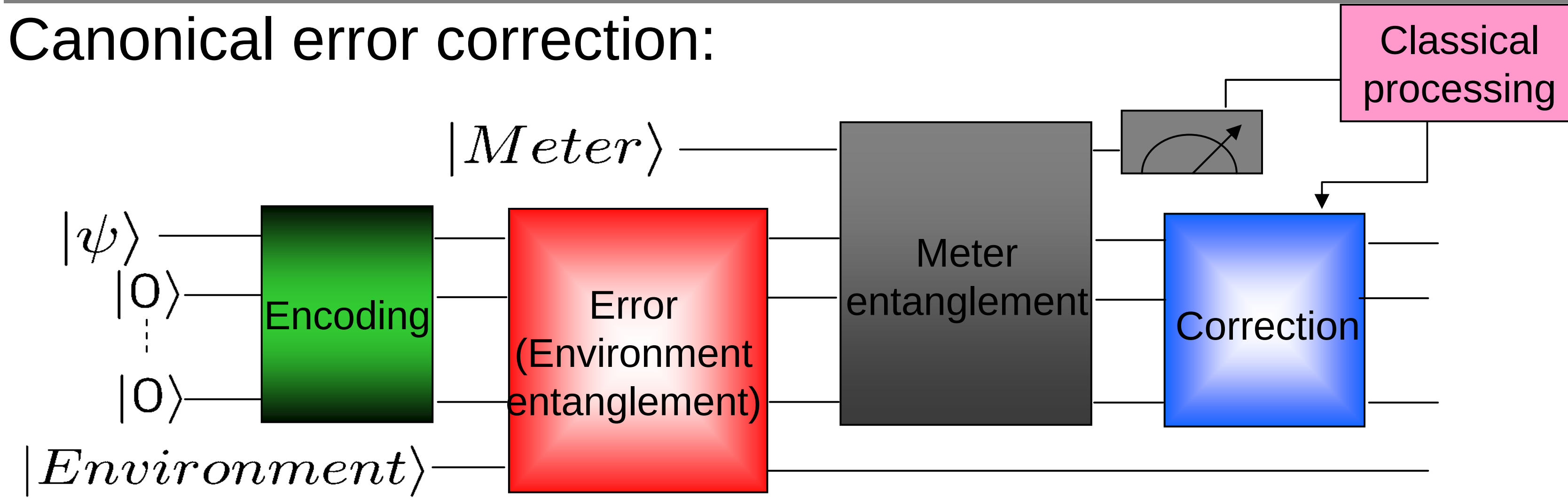
(Compute)

- QC: Hamiltonian feedback
- QEC: unitary gates

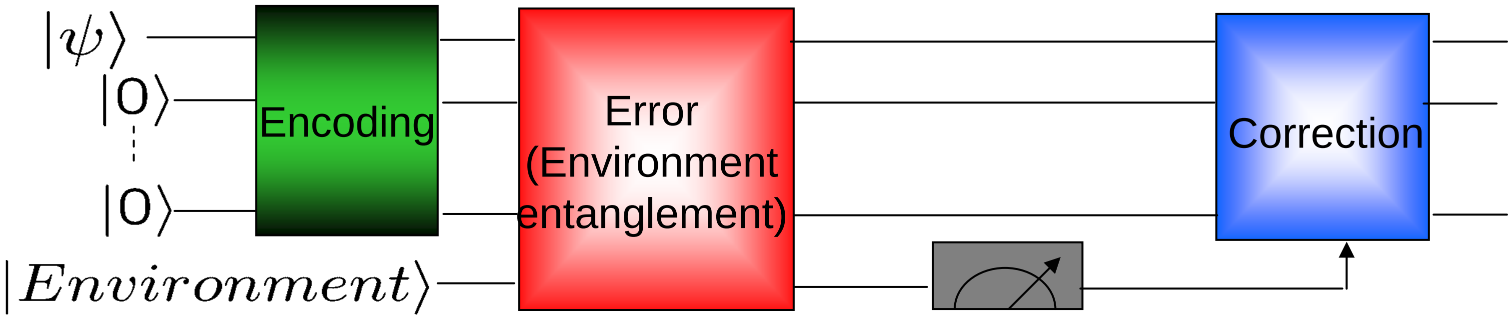
•Ahn, Doherty, Landahl (PRA 65, 042301 (2002))

A different error model

Canonical error correction:



Our scheme: dominant error process: we know when and where the error has occurred, and which one it is.



Continuous detection: master equation

Kraus operators:

$$\Omega_1 = c\sqrt{dt}$$ “Jump” term

$$\Omega_0 = 1 - c^\dagger c dt/2 - iH dt$$ “No-jump” term

Unconditional master equation without feedback

$$\begin{aligned} d\rho &= \Omega_0 \rho \Omega_0 + \Omega_1 \rho \Omega_1 - \rho \\ &= -i[H,\rho]dt + c\rho c^\dagger dt - \frac{1}{2}(c^\dagger c \rho + \rho c^\dagger c)dt \\ &\equiv -i[H,\rho]dt + \mathcal{D}[c]\rho dt. \end{aligned}$$

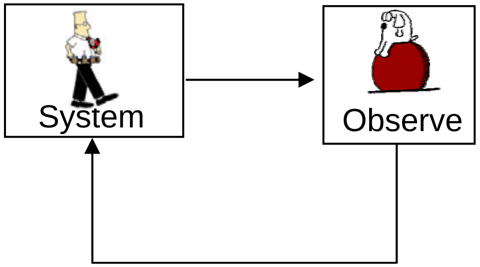
Lindblad form

An error correction protocol must

- Be able to correct errors due to Ω_1
- Find way to get rid of no-jump error due to Ω_0

Continuous detection: adding feedback (jump evolution)

Add feedback via Hamiltonian V that is proportional to the measurement signal: $H_{fb}(t) = \frac{dN(t)}{dt} V$



$$\begin{aligned} dN_c(t)^2 &= dN_c(t) \\ E[dN_c(t)] &= \langle c^\dagger c \rangle_c dt \end{aligned}$$

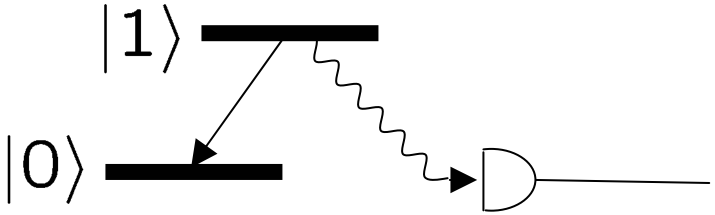
Resulting master equation:

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[e^{-iV} c]\rho$$

Adds unitary to decohering operator

H.M. Wiseman, PRA 49, 2133 (1994), PRA 50, 4428 (1994)

Example: Spontaneous emission



$$c = \frac{X - iY}{2}$$

Kraus operators:

$$\Omega_1 = c\sqrt{dt}$$

$$\Omega_0 = 1 - \frac{c^\dagger c}{2} dt = 1 - \frac{1+Z}{4} dt$$

$$X \equiv \sigma_x$$
$$Y \equiv \sigma_y$$
$$Z \equiv \sigma_z$$
$$XX \equiv \sigma_x \otimes \sigma_x$$

Nontrivial no-jump evolution!

Previous work: Spontaneous emission

- Plenio, Vedral, Knight (PRA **55**, 67 (1997)): 8-qubit code
  - One arbitrary error, no-emission evolution
- Leung et al. (PRA 56, 2567 (1997)): 4-qubit code
  - One spontaneous emission error, no-emission evolution to first order
- Alber et al. (PRL 86, 4402 (2001)): 4-qubit code
  - More specific problem:
    - Spontaneous emission from statistically independent reservoirs
    - Only errors possible are spontaneous emission errors
    - Time and position of each error is known
  - One spontaneous emission error, no-emission evolution
- AWM: 2-qubit code with driving Hamiltonian
  - Alber et al's problem
  - One spontaneous emission error, no-emission evolution

Spontaneous emission: Two-qubit code (Jump evolution)

Codewords:

$$|\bar{0}\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\bar{1}\rangle \equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

These states are +1 eigenstates of XX: XX is stabilizer

Kraus operators:

$$\Omega_j = \sqrt{\kappa_j dt}(X_j - iY_j) \equiv \sqrt{\kappa_j dt}a_j$$

These operators correspond to correctable errors: they obey the condition

$$\langle \psi_\mu | \Omega_j^\dagger \Omega_j | \psi_\nu \rangle = \Lambda_j \delta_{\mu\nu}$$

for orthogonal codewords ψ_μ, ψ_ν.

PRA 55, 900 (1997)

These operators can be corrected by applying a suitable unitary;
A suitable unitary can be generated by a feedback Hamiltonian;
$\Rightarrow$ Errors can be corrected via a feedback Hamiltonian

$$H_{fb} = \frac{dN_1}{dt}(XI - ZX) + \frac{dN_2}{dt}(IX - XZ)$$

Spontaneous emission: What about getting rid of the no-jump evolution?

- For this two-qubit system, the no-jump Kraus operator is

$$\Omega_0 = II\,(1-(\kappa_1+\kappa_2)dt) - \kappa_1 dt ZI - \kappa_2 dt IZ - iHdt$$

Add a Hamiltonian?

- First guess: $H = idt(\kappa_1 ZI + \kappa_2 IZ)$?
  - No because H is then not Hermitian!

- Better: $H = -(\kappa_1 YX + \kappa_2 XY)$

Then

$$\Omega_0 = II\,(1-(\kappa_1+\kappa_2)dt) - \kappa_1 dt ZI(II - XX) - \kappa_2 dt IZ(II - XX)$$

Annihilates codespace

Putting it all together

The final master equation:

$$d\rho = \Omega_0 \rho \Omega_0^\dagger - \rho + dt \sum_{j=\{1,2\}} \kappa_j U_j c_j \rho c_j^\dagger U_j^\dagger$$

Acts trivially on code subspace

U_j is unitary correcting for error c_j

Evolution preserves code subspace.

Addition of Hamiltonian gives Kraus operator that preserves code subspace while allowing small redundancy!

General case: (very quick) stabilizer review

- Define Pauli group as

$$P_n = \{\pm 1, \pm i\} \otimes \{I, X, Y, Z\}^{\otimes n}$$

- Given 2^n-dim Hilbert space, a stabilizer S is a subgroup of 2^{n-k} commuting elements of the Pauli group

- Properties of codespace associated with this stabilizer:
  - It is the +1 simultaneous eigenspace of the stabilizer group:
    $s_i|\psi_\mu\rangle = |\psi_\mu\rangle$ for all $|\psi_\mu\rangle$ in codespace
    and all $s_i \in S$
  - It encodes k logical qubits in n physical qubits

D. Gottesman, PRA 54, 1862 (1996)

General error and unraveling

- Code with stabilizer S:

$$S|\psi_\mu\rangle = |\psi_\mu\rangle \text{ for all } |\psi_\mu\rangle \text{ in codespace}$$

- Different unravelings of master equation parametrized by γ:

$$\Omega_j = (c_j + \gamma)\sqrt{dt}$$

- Recall that the condition $\langle\psi_\mu|\Omega_j^\dagger\Omega_j|\psi_\nu\rangle = \Lambda_j\delta_{\mu\nu}$ determines whether the error is correctable, and note

$$\Omega_j^\dagger\Omega_j = aI + bD$$

a, *b* complex
D Hermitian

- Familiar sufficient condition for stabilizer code:
Stabilizer anticommutes with traceless part of $\Omega_j^\dagger\Omega_j$

{*S*,*D*} = 0

If this is true, feedback can correct the state.

General unraveling: no-jump evolution

- No-jump evolution is given by

$$\begin{aligned}\Omega_0 &= 1 - \frac{1}{2}\Omega_i^\dagger \Omega_i dt - \frac{\gamma}{2}(c - c^\dagger)dt - iHdt \\ &\equiv \chi 1 - D dt - \frac{\gamma}{2}(c - c^\dagger)dt - iHdt\end{aligned}$$

Assume

$\{S,D\} = 0$

- Choose driving Hamiltonian

$$H = \tfrac{i}{2}DS + \tfrac{i\gamma}{2}(c - c^\dagger)$$

- Check: Hermitian? Yes, because $\{S,D\} = 0$.
- Total Ω_0 evolution is now

$$\begin{aligned}\Omega_0 &= \chi 1 - \frac{1}{2}D - i(\frac{i}{2}DS) \\ &= \chi 1 - \frac{1}{2}D(1 - S)\end{aligned}$$

Annihilates codespace!

Generalization to n qubits:

Let $\{c_j\}$ be a set of errors such that c_j (with associated operator D_j) acts on the j^{th} qubit alone.

Since D_j is traceless, it is always possible to find some other Hermitian traceless one-qubit operator s_j such that

$$\{s_j, D_j\} = 0.$$

Then we may pick the stabilizer group by choosing the single stabilizer generator

$$S = s_1 \otimes \cdots \otimes s_n.$$

• c_j errors are correctable

Now choose the driving Hamiltonian

$$H = \sum_j \frac{i}{2} D_j S + \frac{i|\gamma_j|}{2}(c_j - c_j^\dagger)$$

• No-jump evolution corrected!

Encodes n-1 qubits in n!

Ahn, Wiseman, Milburn PRA 67, 052310 (2003)

Multiple channels

Previously we assumed only one perfectly measured channel per qubit. Let us now assume there is more than one channel we can measure. Can we still perfectly correct?

Choose stabilizers
$$\begin{aligned} S_1 &= XXXX \\ S_2 &= ZZZZ \end{aligned}$$

(This is just the four-qubit error correction code!)

Decompose an error operator D as $D = \vec{d} \cdot \vec{\sigma}$

It is always possible to find some S_j such that $\{S_j, \sigma_j\} = 0$

Set $H = \sum_j i(d_j \sigma_j) S_j$

Then
$$\begin{aligned} \Omega_0 &= a1 - Ddt - iHdt \\ &= a1 - \sum_j d_j \sigma_j (1 - S_j) \end{aligned}$$

Encodes n-2 qubits in n!

Ahn, Wiseman, Jacobs, to appear in PRA (2004)

…but life isn't perfect

- What if we don't know the error rate perfectly?
  - For error rate κ, using $\kappa(1+\varepsilon)$ in the analysis instead of κ results in a state that differs to first order in ε.

- Detection inefficiency η:
  - If detection efficiency is a more realistic $\eta \leq 1$

    the jump master equation acquires an extra term

$$\dot{\rho}_{\text{eff}} = (1-\eta)\sum_j (c_j \rho c_j^\dagger - U_j c_j \rho c_j^\dagger U_j^\dagger)$$

  - This results in exponential decay of coherence of the subspace!
    - This is a property of any continuous error-correction protocol that relies on correcting errors instantaneously after they occur
    - Tradeoff between computational complexity and robustness to error

Summary

- Possible to understand a particular variant of quantum control as quantum error correction
- Showed error correction protocol with following properties:
  - Can correct any single qubit detected errors
  - Requires only n physical qubits to encode n-1 logical qubits
  - Can show that this protocol allows for universal quantum computation
- More questions
  - A compromise between computational complexity and robustness? How to characterize?
  - Using control theory techniques to further explore problem of stabilizing subspace?
  - Qudits?