Quantum Computing with Very Noisy Gates

Produced with pdflatex and xfig

- Fault-tolerance thresholds in theory and practice.
- Available techniques for fault tolerance.
- A scheme based on the $[[4,2,2]]$ code.
- Resource requirements.

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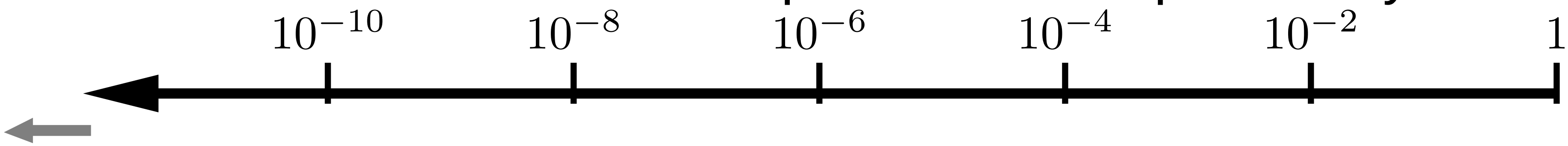
Fault Tolerant Quantum Computing

Fault-Tolerance Threshold Theorem: *Given: Noisy qubits and gates. If the error rates are sufficiently low, then it is possible to efficiently process quantum information arbitrarily accurately.*
Shor (1995) [1, 2], Kitaev (1996) [3], Aharonov&Ben-Or (1996) [4], Knill&Laflamme&Zurek (1996) [5], . . . Gottesman&Preskill (1999), . . . Steane (2002) [6], . . . Knill (2004) [7], Reichardt (2004) [8]

- What is required of “noisy qubits and gates”?
- What is “sufficiently low”?
- What is “efficient”?

Error Thresholds: Proofs and Estimates

- Error thresholds in model-dependent “error probability”.



Correlations $\to 1$. Clemens&Siddiqui&Gea-Banacloche (2004) [9]

Adversarial, quasi-independent.

Knill&Laflamme&Zurek (1996) [5], Terhal&Burkard (2004) [10], Alicki (2004) [11]

Adversarial, quasi-independent, probabilistic Pauli.

Aharonov&Ben-Or (1996) [4], Knill&Laflamme&Zurek (1996) [5]

Depolarizing errors. Gottesman&Preskill (1999)

Steane (2002) [6]

Knill (2004) [7], Reichardt (2004) [8]

Detected errors. Knill (2003) [12]

Unintended Z-measurements.

Knill (2002)

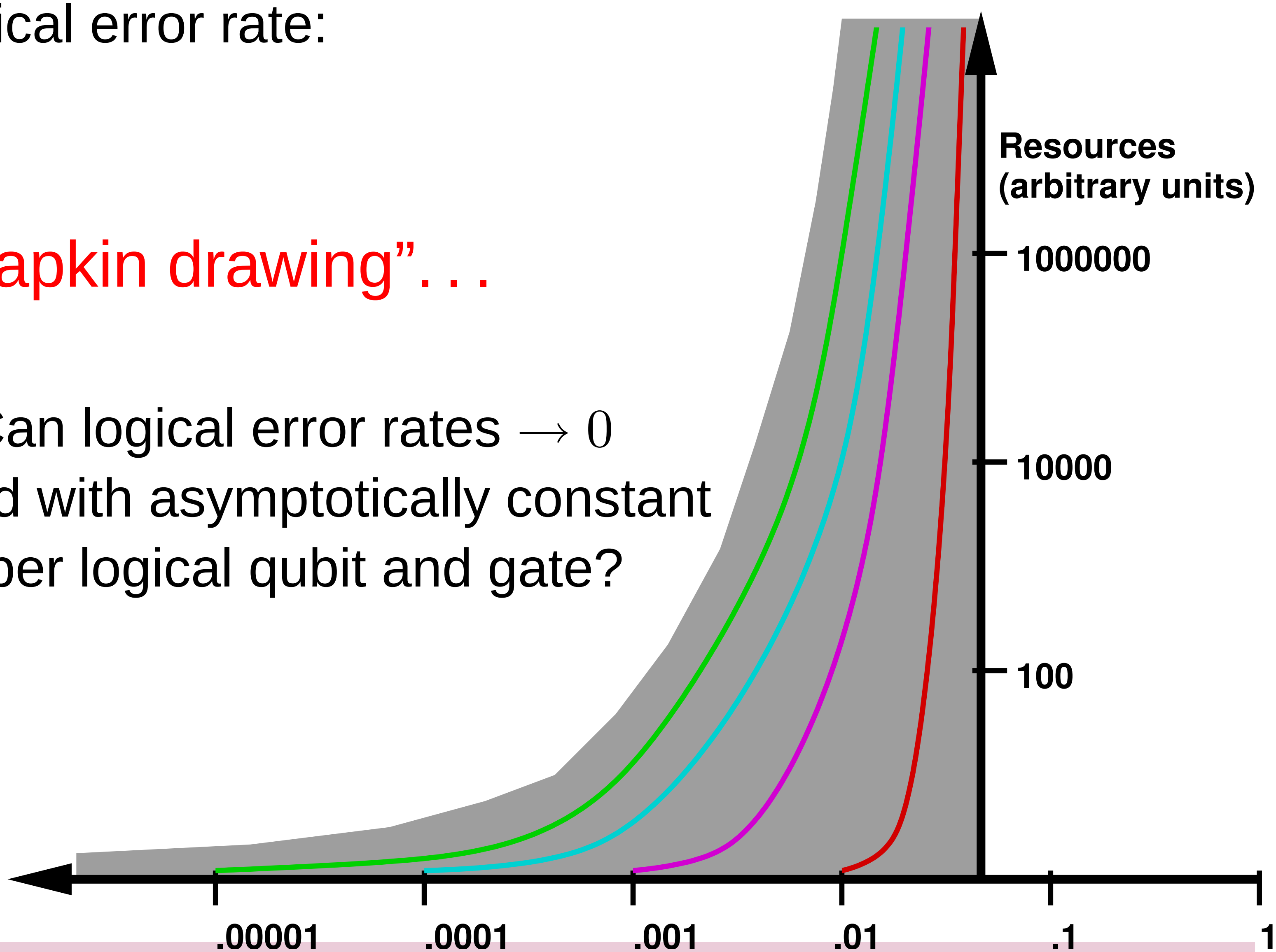
The Setting

- Physical qubit engineering process.
  - Minimize noise in classical control fields.
    E.g. by proper shielding.
  - Reduce systematic errors in gates.
    E.g. by self-correcting pulse sequences.
  - Take advantage of available noiseless subsystems.
    E.g. decoherence free subspaces.
  - Balance noisy behavior.
    E.g. Improve measurements if **cnot**'s have low noise.
  - Take advantage of error-detection if possible.
    E.g. by detecting emitted photons.

- To be considered here: Model-independent methods.
  - further physical engineering is relatively expensive.
  - errors are generic, with no known exploitable biases.

Error Thresholds: Theory and Practice

- Resources per logical qubit and gate required to achieve a given logical error rate:



This is a “napkin drawing”. . .

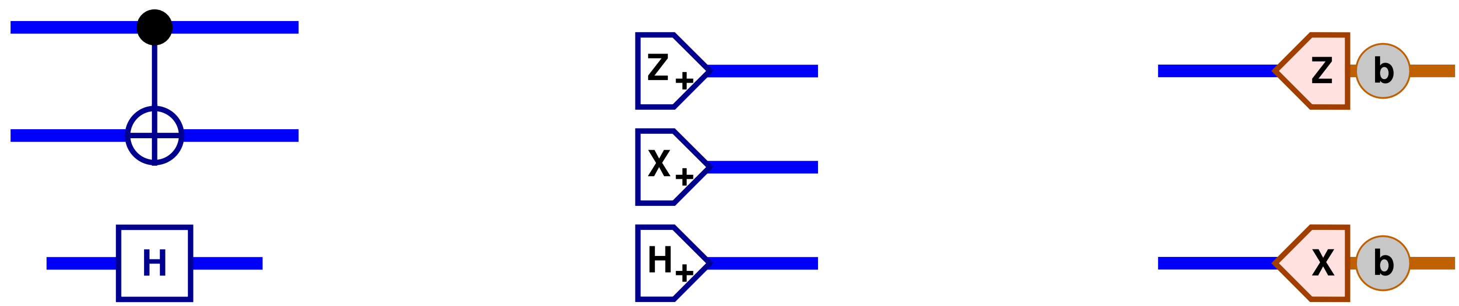
- Problem: Can logical error rates $\rightarrow 0$ be achieved with asymptotically constant resources per logical qubit and gate?

Structural Assumptions

Physical resources:

- Arbitrarily many "physical" qubits can be called on.

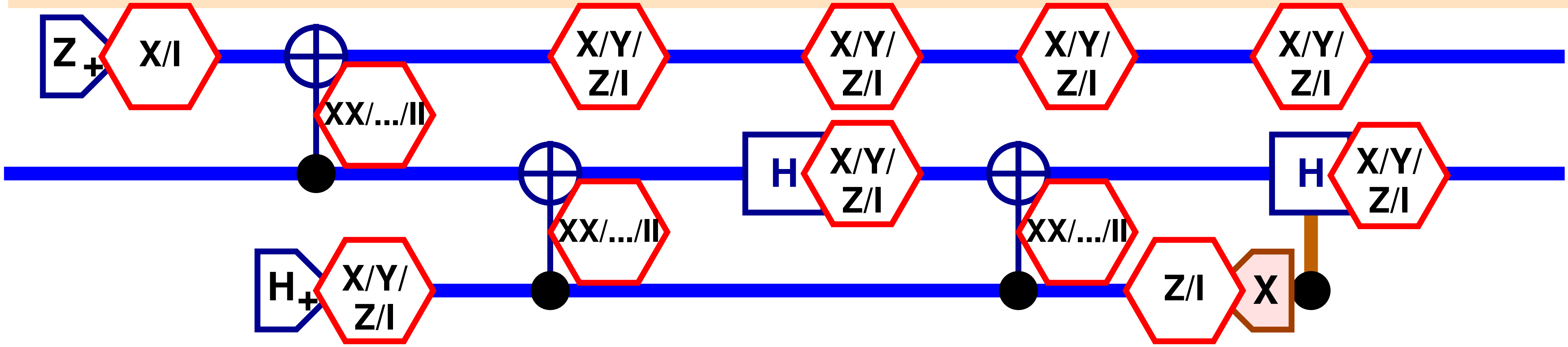
Local control capabilities:



Global control capabilities:

- Massive parallelism or no memory error.
- Negligible classical computation latency.
- Negligible quantum communication latency (for non-local two-qubit gates).

Error Models



- Independent, probabilistic Pauli errors:
  - The $|e_{\mathbf{p}}\rangle$ are orthogonal.
  - $\||e_{\mathbf{p}}\rangle|^2 = \prod_i e_i(p_i)$, where e_i depends only on the gate type.
- Justification?
  - Stabilizer code implementations imply short lifetimes of unwanted coherences between Pauli errors.
  - Random Pauli pulses can further reduce these lifetimes.
  - Correlations are usually local.

Independent Depolarizing Error Models

- Each operation's errors are uniformly random Pauli errors.
  - $|0\rangle$ preparation with noise:
    Prepared state is $\{(1-e_p) : |0\rangle, e_p : \sigma_x|0\rangle\}$.
  - Measurement with noise:
    $|\psi\rangle \rightarrow \{(1-e_m) : \mathbb{1}, e_m : \sigma_x\}$ before σ_z measurement.
  - No operation (memory) with noise:
    $\{(1-e_n) : \mathbb{1}, e_n/3 : \sigma_x, e_n/3 : \sigma_y, e_n/3 : \sigma_z\}H$.
  - Hadamard with noise:
    $\{(1-e_h) : \mathbb{1}, e_h/3 : \sigma_x, e_h/3 : \sigma_y, e_h/3 : \sigma_z\}H$.
  - Cnot with noise:
    $\{(1-e_c) : \mathbb{1}, e_c/15 : {\sigma_x}^{(1)}, \ldots, e_c/15 : {\sigma_z}^{(1)}{\sigma_z}^{(2)}\}\textbf{cnot}^{(12)}$.
- Agnostic choice for $e_{p,m,h,c}$?

$$e_c = \epsilon, \quad e_h = \frac{12}{15}\epsilon, \quad e_p = \frac{4}{15}\epsilon, \quad e_m = \frac{4}{15}\epsilon, \quad e_n \leq e_h.$$

Fault-Tolerance Methods

- Use of stabilizer codes to define logical qubits. Shor (1995) [1], Steane (1995) [13]
- Transversal encoded Clifford-Pauli group operations. Shor (1996) [2]
- Teleported gates. Gottesman&Chuang (1999) [14]
- Non-destructive and fault-tolerant syndrome measurements. Shor (1996) [2], Steane (1999) [15]
- Concatenation. Aharonov&Ben-Or (1996) [4, 16], Knill&Laflamme (1996) [17], . . .
- Teleported error-correction. Knill (2003) [12]
- Fault-tolerant postselected quantum computation. Knill (2004) [18]
- Bounded-error state preparation via decoding. Knill (2004) [18]
- Purification of “magic” states. Bravyi&Kitaev (2004) [19], Knill (2004) [18]
- Error-correction with likelihood tracking. In progress.
- { Dynamically encoded logical qubits.
  Sparse codes. } Two thoughts. . .

The Clifford-Pauli Group

- Pauli matrix notation.

$$I = \mathbb{1}, X = \sigma_x, Y = \sigma_y, Z = \sigma_z$$
$$[IXIYI] = \sigma_x^{(2)}\sigma_y^{(4)} = \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \sigma_y \otimes \mathbb{1}$$

  $\mathcal{P}_n$ is the set of the ± 1 Pauli products on n qubits.

- The Clifford-Pauli group:

$$\mathcal{N}_n = \left\{ U \,|\, U\mathcal{P}_n U^\dagger = \mathcal{P}_n, U \text{ is unitary} \right\}$$

- Generators of $\mathcal{N}_n$: **cnot**, H, $e^{-iZ\pi/4}$

- **Theorem.** Any quantum computation using Z-eigenstate preparation, operators in $\mathcal{N}_n$, Z-measurements and feedforward can be efficiently classically simulated.

Gottesman (1997) [20]

Power of Clifford-Pauli Operations

- The CSS operations, $\mathcal{CSS}$:
  Preparation of $|\mathtt{o}\rangle$ and $|+\rangle$, **cnot**, Measurement of X and Z.
  - CSS operations suffice for encoding/decoding CSS codes.

- Universal quantum computation is possible with $\mathcal{CSS}$, H and $|\pi/8\rangle$-preparation.

$$\text{QC} = \overbrace{\mathcal{CSS} + \underbrace{\text{“}\epsilon\text{”}}_{H}}^{\subseteq\mathcal{N}} + \underbrace{\text{“}\delta\text{”}}_{|\pi/8\rangle}.$$

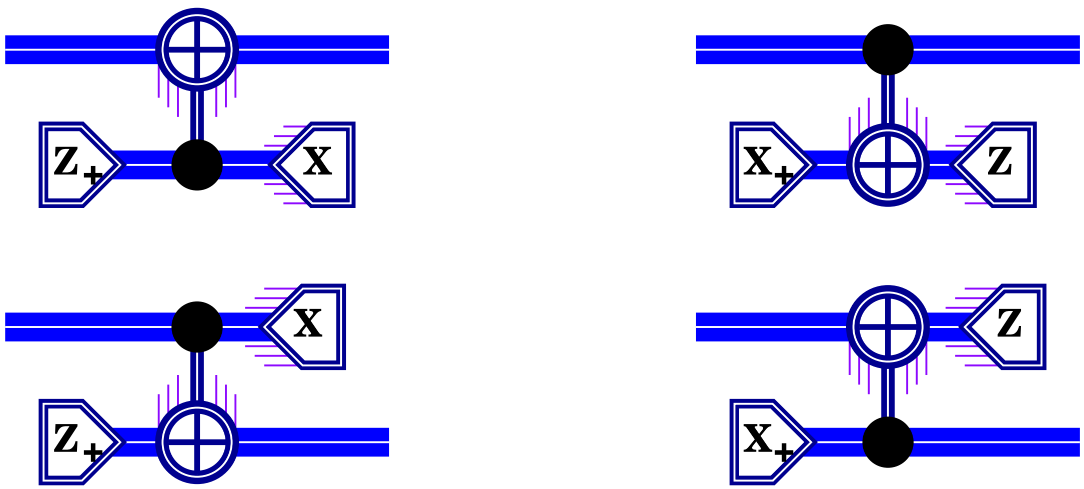
- A fault-tolerant computation strategy:
  1. Implement a fault tolerant CSS computer, i.e. arbitrarily accurate logical $\mathcal{CSS}$ with feedforward.
  2. $+\text{“}\epsilon\text{”} + \text{“}\delta\text{”}\ldots$

     $+\text{“}\delta\text{”}$: $|\pi/8\rangle$ purification using good $\mathcal{CSS} + \text{”}\epsilon\text{”}$
     Bravyi&Kitaev (2004) [19], Knill (2004) [18]

- F.-t. $\mathcal{CSS}$ and ($|\pi/8\rangle$ error) $\leq$ ($|\mathtt{o}\rangle$, $|+\rangle$ error) $\Rightarrow$ f.-t. QC?

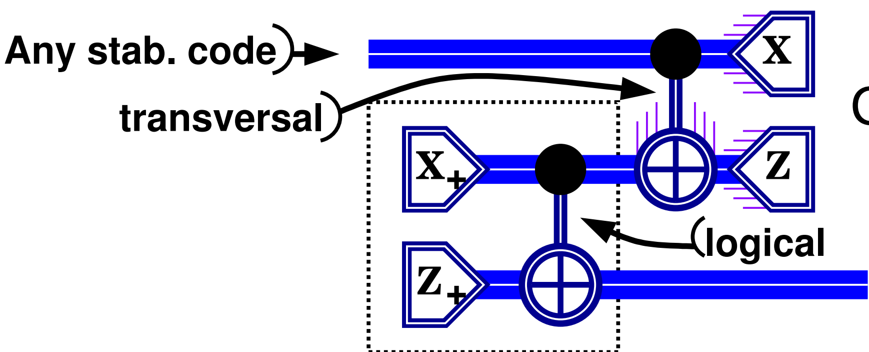
Syndromes and Error Tracking

- Stabilizer code: Eigenspace of commuting Pauli products $\mathcal{S}$.
- Error tracking by nondestructive syndrome measurements.
  - $\mathcal{S}$-syndrome of a joint $\mathcal{S}$-eigenstate $|\psi\rangle$: The $\mathcal{S}$ eigenvalues of $|\psi\rangle$.
- $\mathcal{S}$-measurement by encoded noops.



Cat state schemes
Shor (1996) [2]
Steane (1999) [15]

1-qubit teleportation
Zhou&Leung
&Chuang (2000) [21]



Complete syndrome.

Fresh qubits.

Full teleportation
Knill (2003) [12]

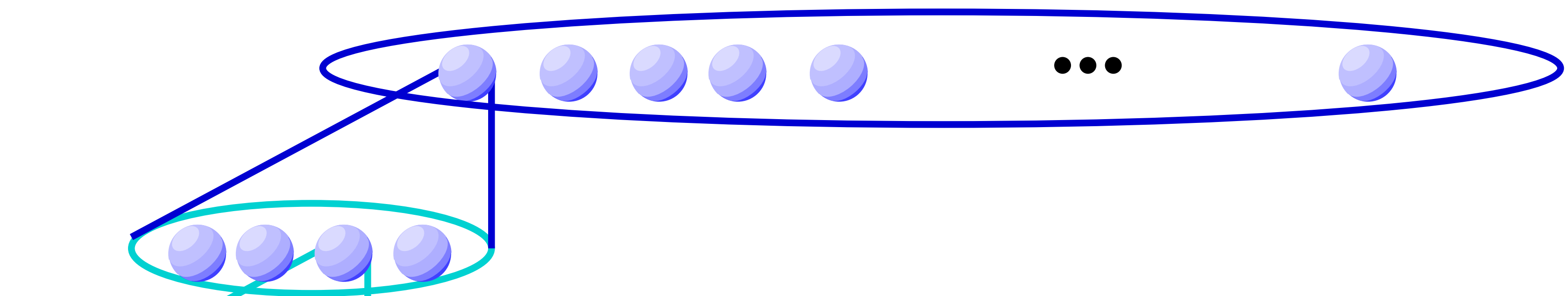
Postselected Fault-Tolerant Quantum Computers

- State preparation $+$ teleportation $\rightarrow$ quantum computation.
  Gottesman&Chuang (1999) [14]
- State preparation need not be deterministic.
- Postselected quantum computers.
  - Can execute any of the basic operations, but
  - an operation may fail, possibly destructively.
  - If an operation fails, this is announced.

  ...exponentially small success probability (not 0) is possible.
- A postselected QC is fault-tolerant if
  success $\rightarrow$ negligible probability of error.
- A postselected f.-t. QC only needs to detect errors.
- Does postselected f.-t. QC imply f.-t. QC?

Toward Unconditional Quantum Computation

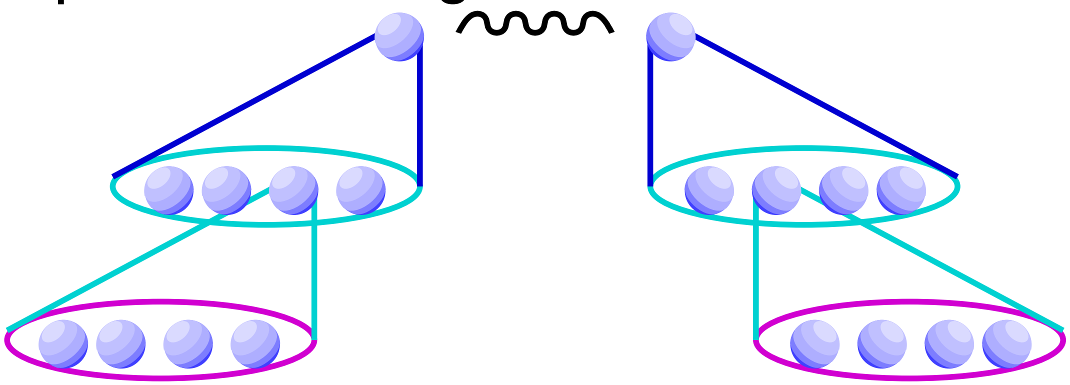
- Problem: The states needed for f.-t. QC must be disturbed by well-bounded local errors only.
- A solution with postselected f.-t. QC:
  1. Implement postselected f.-t. QC with logical qubits based on a small concatenated block code.
  2. Use this to prepare the desired state in encoded form.
  3. Decode the block code through all levels.
  4. Accept the state if no errors are detected in decoding.



E.g. Stabilizer $[XXXX], [ZZZZ], [IIZZ]$,
Logicals $X_L = [XXII]$, $Z_L = [ZIZI]$ (Knill (2004) [18])

Threshold Analysis

- Combine teleported error-detection with concatenated 4-qubit codes.
- Key step: Preparation of logical Bell states.

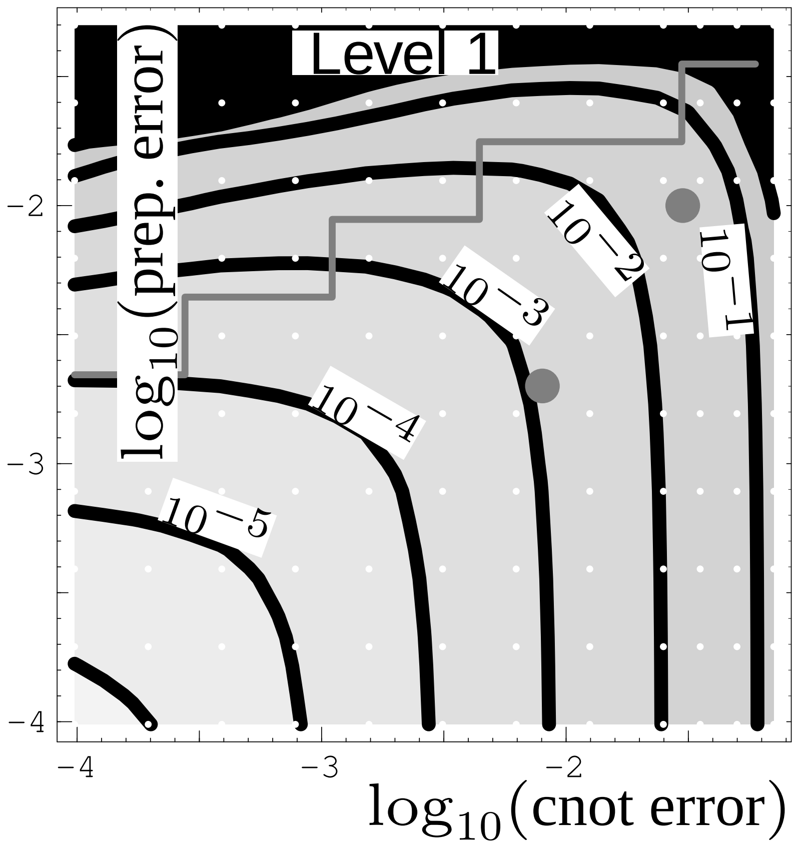


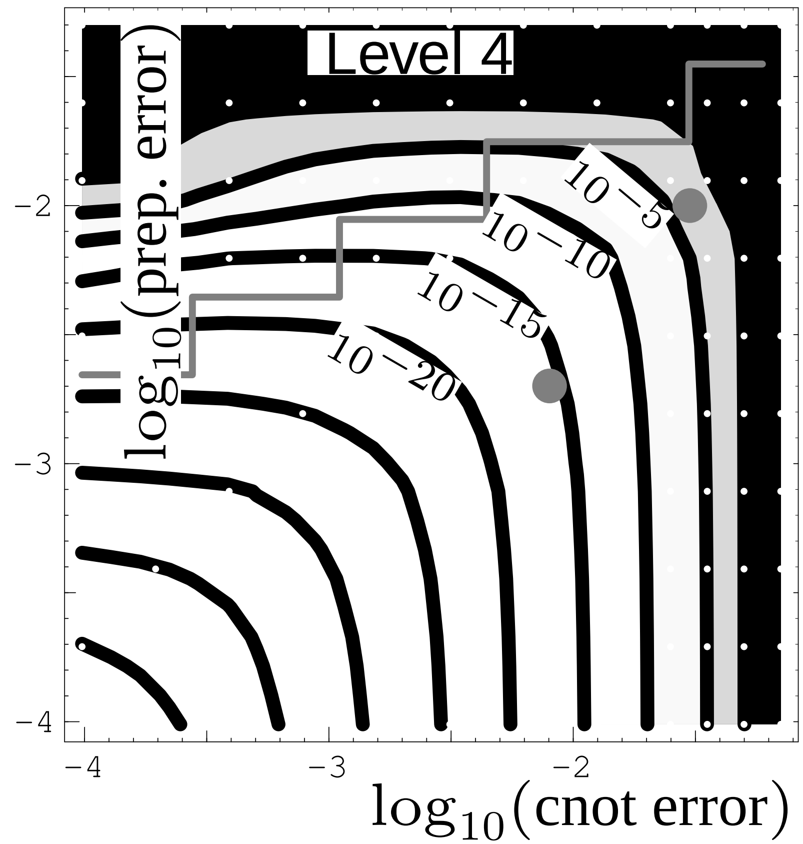
  Goal: $\sim$ Error independence between the Bell halves.

- Actually: Close to independent.
  Analysis: Heuristically "bound" with an independent model.

Logical Error Rates

- Conditional error rates by computer-assisted heuristics.





- Examples.

cnot-error 3%,	prep./meas.-error 1%:	Max. err. $2.7 * 10^{-5}$ at 256 qubits/block
$.8\%$	$.2\%$	$5.3 * 10^{-7}$ at 16 qubits/block.

The [[4,2,2]] Code

- Improve efficiency: Encode multiple qubits.
  Add error correction.

- $[[4,2,2]]$ code. Stabilizer: $[XXXX], [ZZZZ]$.
  Logical ops: $X_L = [XXII], Z_L = [ZIZI]$
  $X_S = [IXIX], Z_S = [IIZZ]$.

- Some properties of the $[[4,2,2]]$ code.
  - For syndrome $(+1,+1)$, the following are logical states:

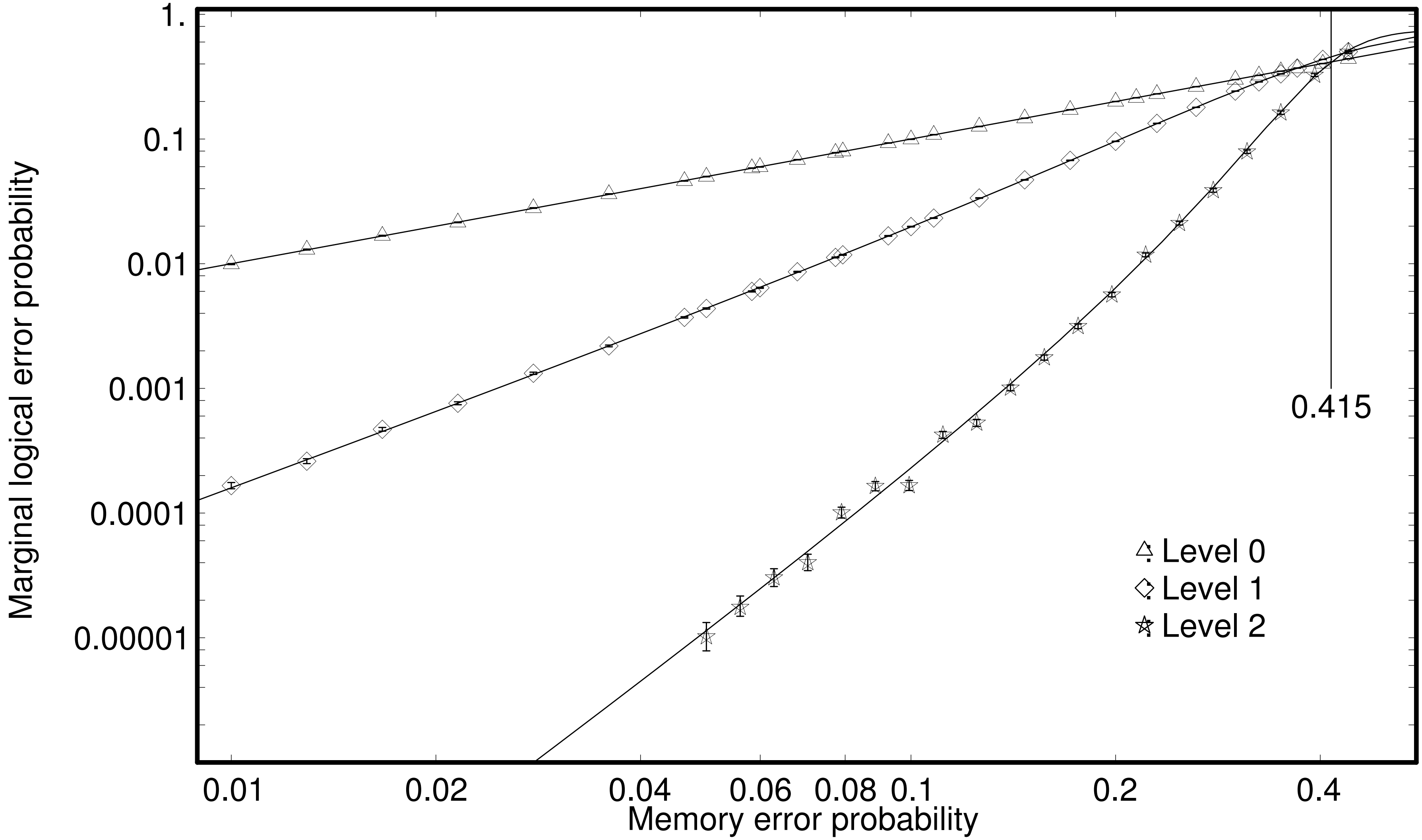
$$\frac{1}{\sqrt{2}}\left(|0000\rangle_{1234} + |1111\rangle_{1234}\right), \frac{1}{\sqrt{2}}\left(|++++\rangle_{1234} + |----\rangle_{1234}\right),$$

$$\frac{1}{2}\left(|00\rangle_{12} + |11\rangle_{12}\right)\left(|00\rangle_{34} + |11\rangle_{34}\right), \frac{1}{2}\left(|00\rangle_{13} + |11\rangle_{13}\right)\left(|00\rangle_{24} + |11\rangle_{24}\right)$$

  - The concatenation $[[4,2,2]] \circledast [[4,2,2]]$ is a $[[16,4,4]]$ code.
  - The logical entanglement $[[4,2,2]] \leftrightsquigarrow [[4,2,2]]$ is a Bell state between a qubit and a logical qubit in the [[7,1,3]] code.

[[4,2,2]]: Cond. Logical Error with Detection

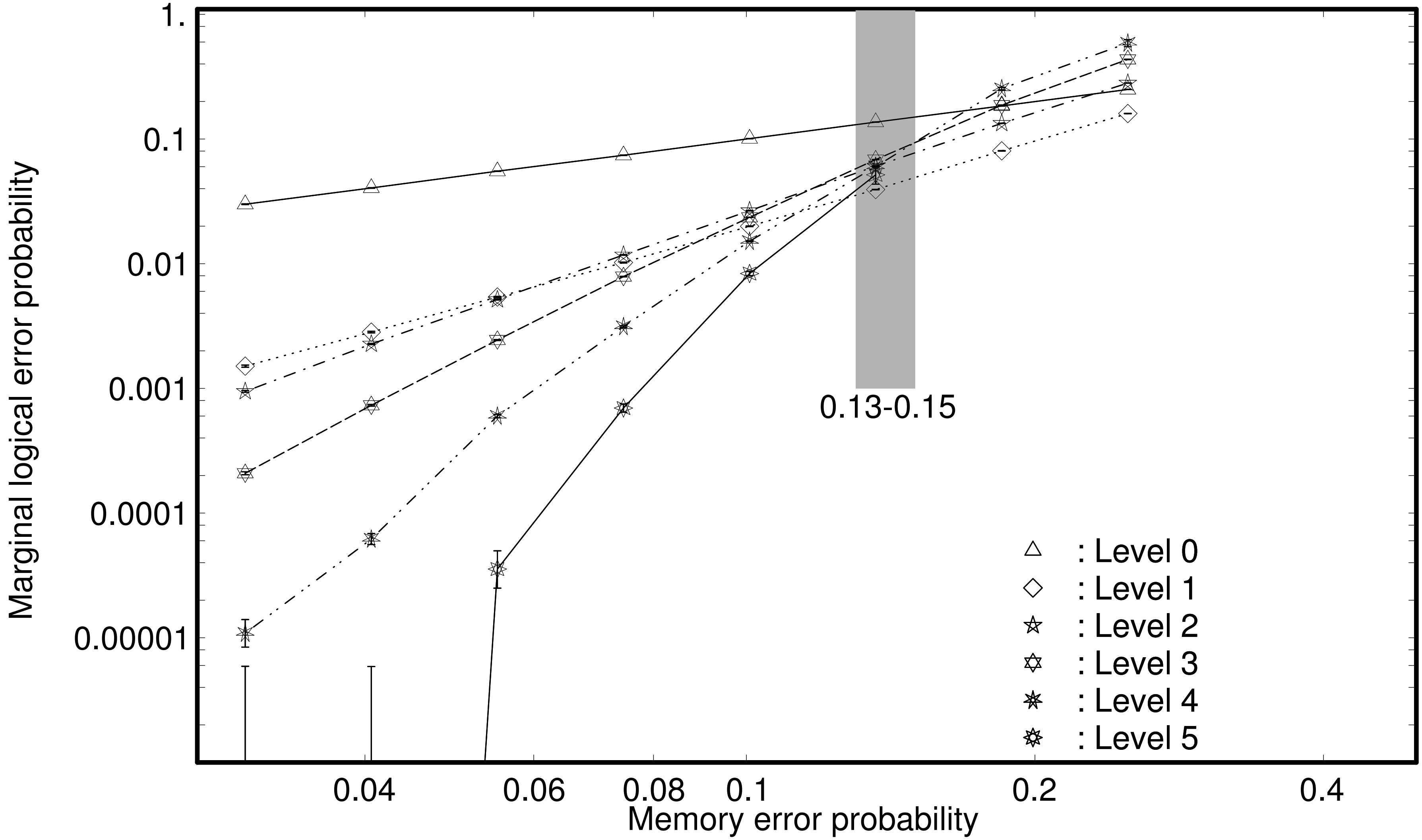
Conditional logical memory error, no error correction, by direct simulation.



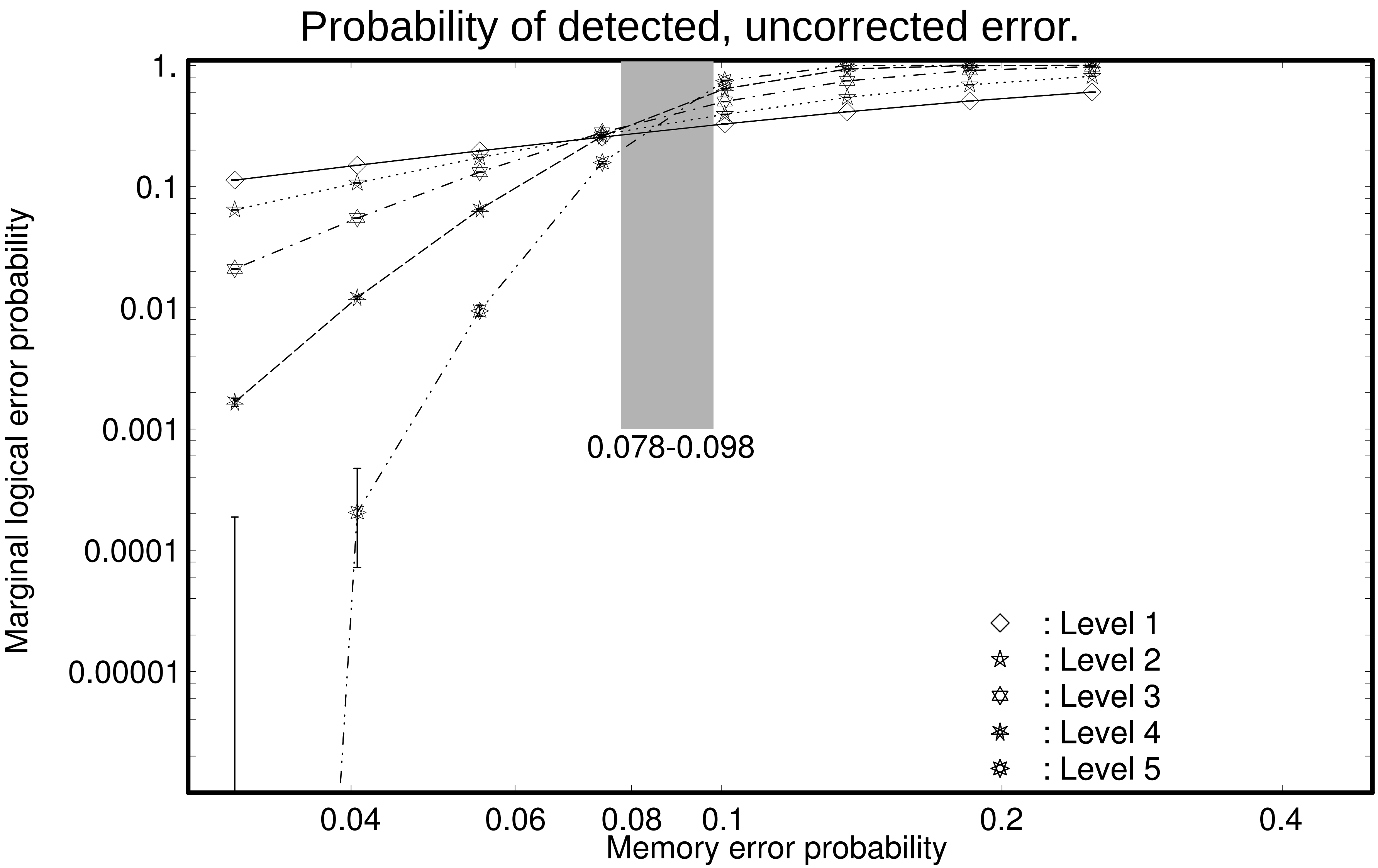
The curves are analytical.

[[4,2,2]]: Cond. Logical Error with Correction

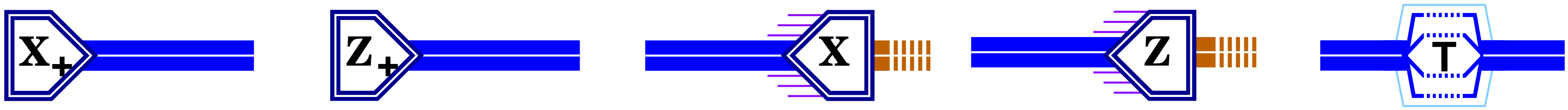
Conditional logical memory error, with error correction, by direct simulation.



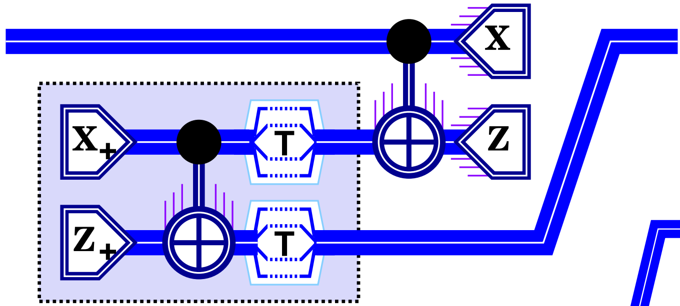
[[4,2,2]]: Detected Error Probability



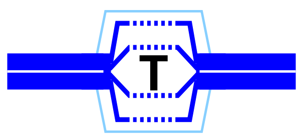
Recursive Bell State Preparation



Logical qubits teleportation:



Implementations:



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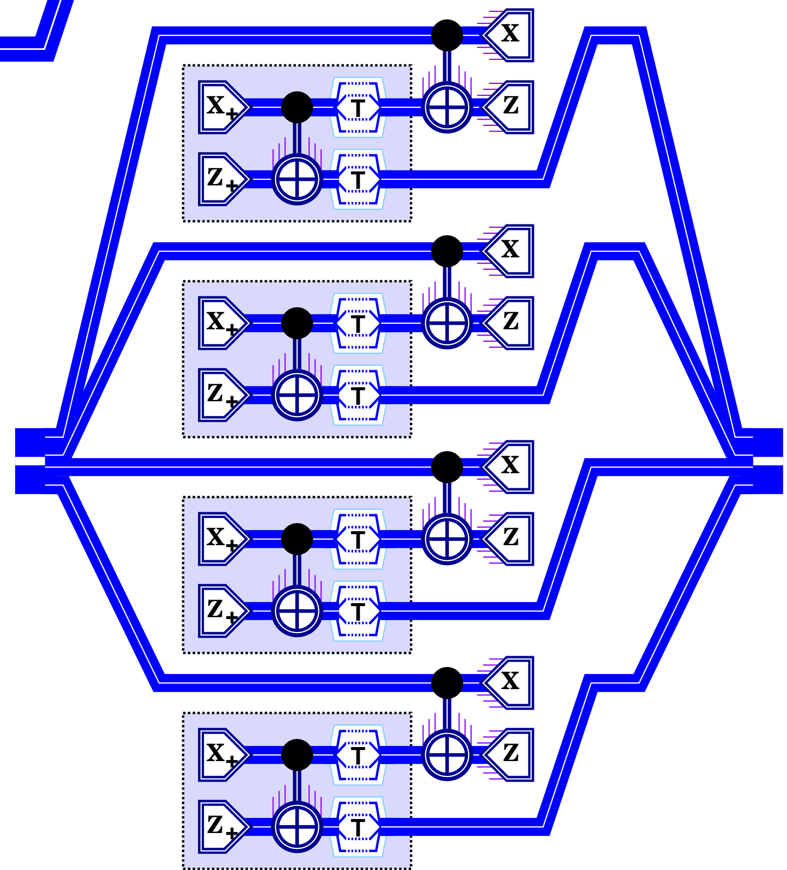
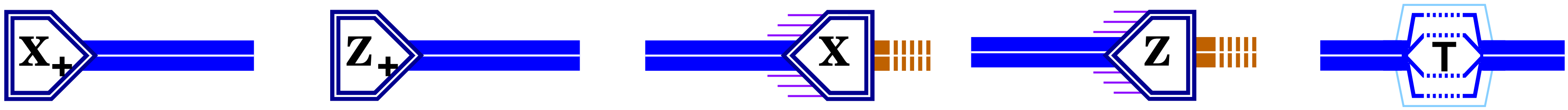
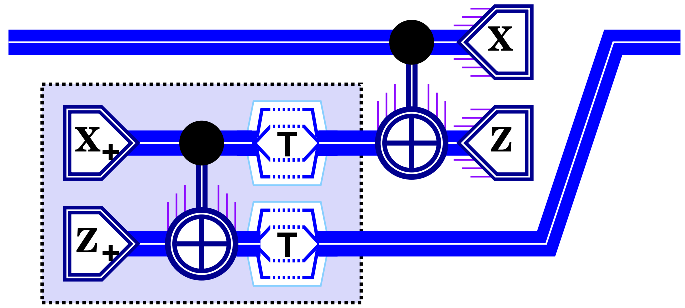


Recursive Bell State Preparation



Logical qubits teleportation:



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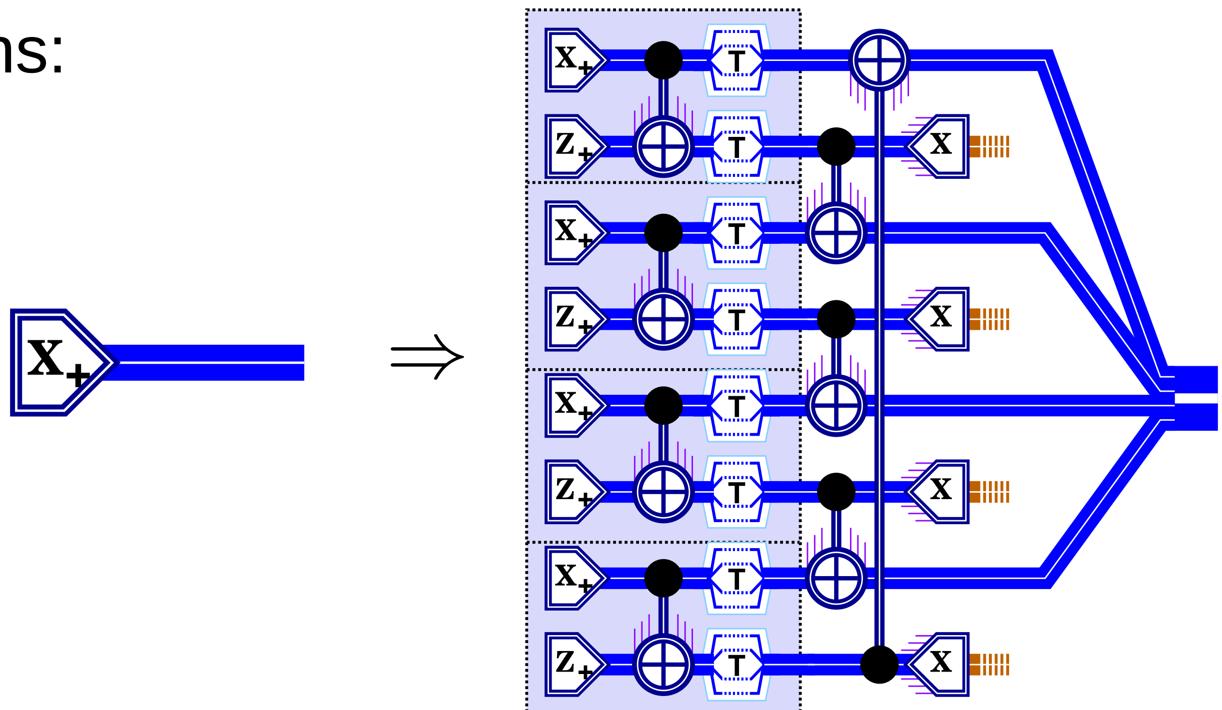

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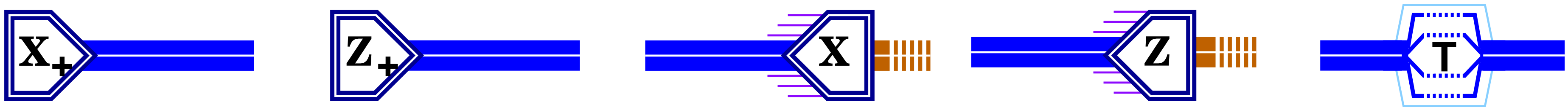
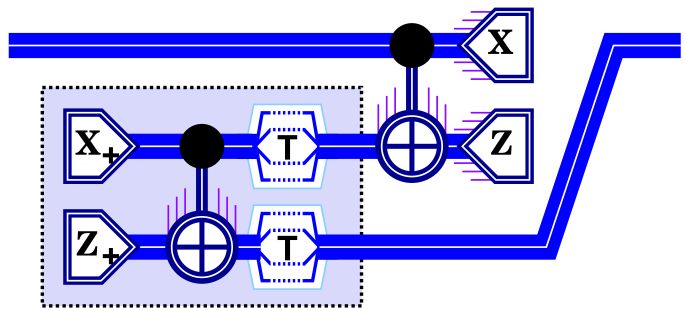


Logical qubits teleportation:



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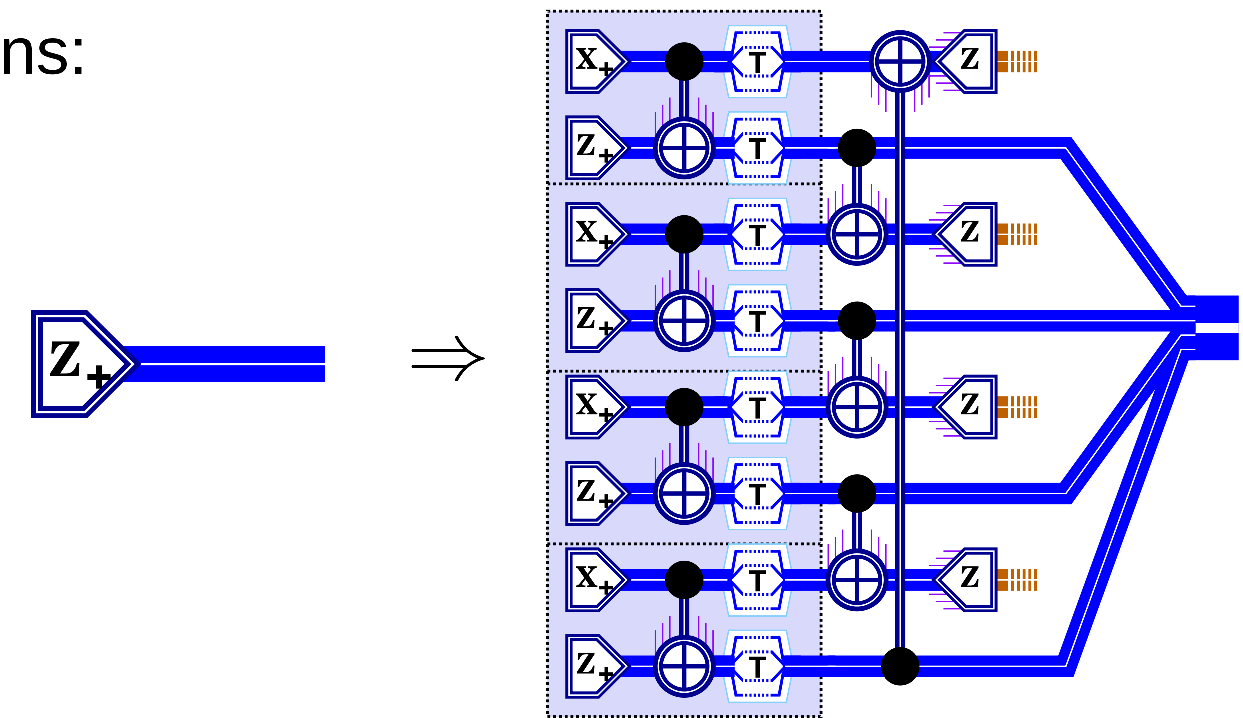
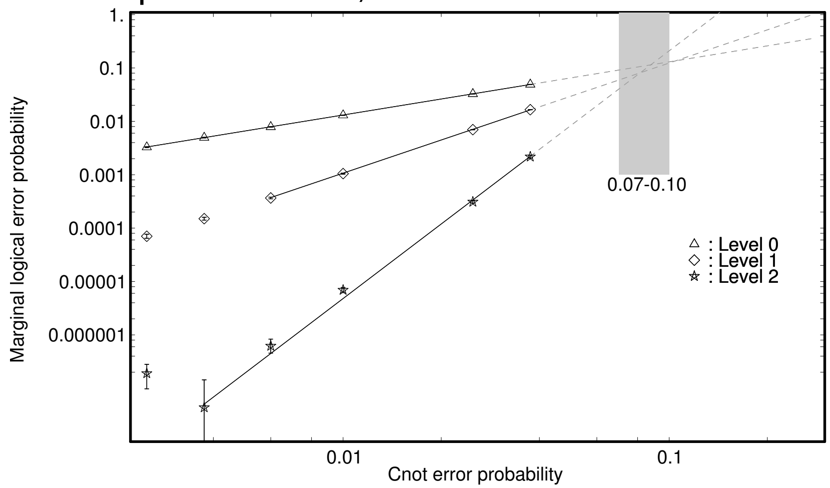
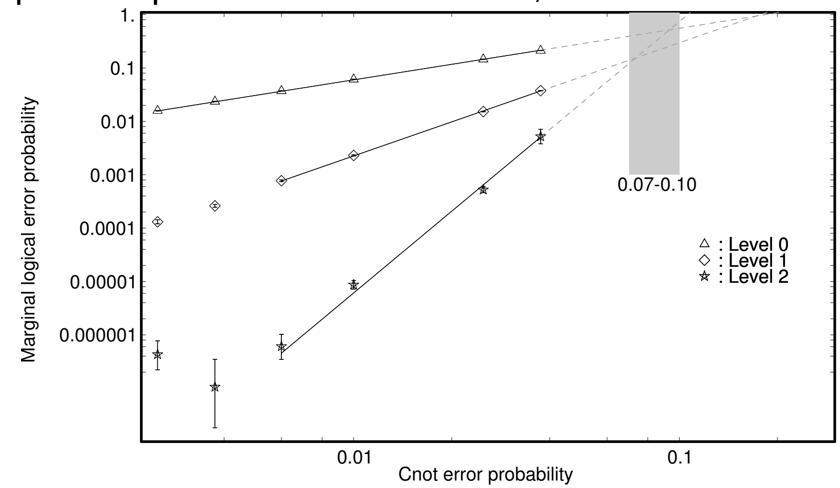


Simulations of Error-Detecting Behavior

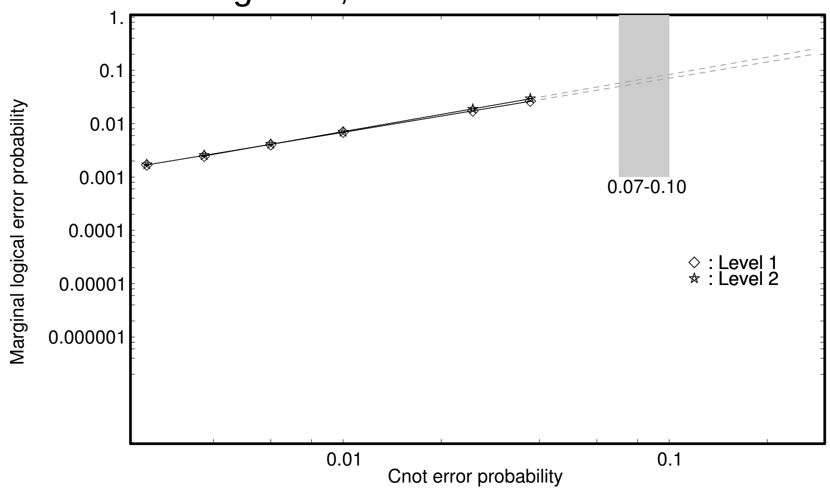
Preparation error, conditional on success.



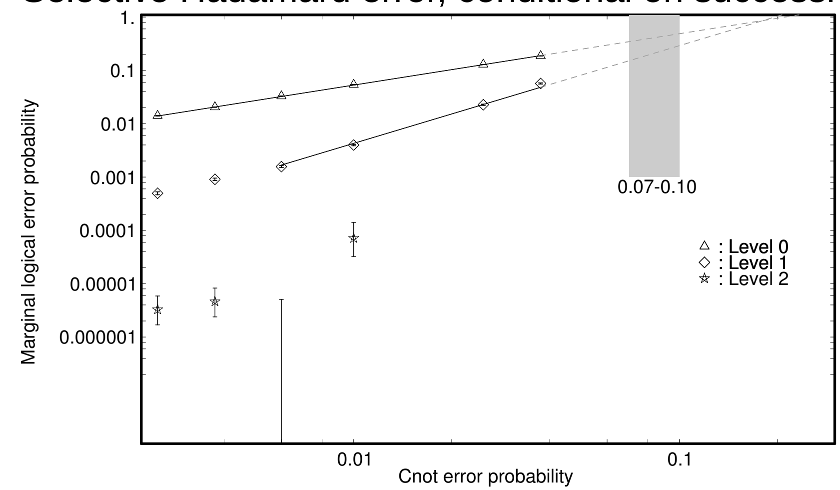
Two-qubit teleportation and cnot error, conditional on success.



Decoding error, conditional on success.

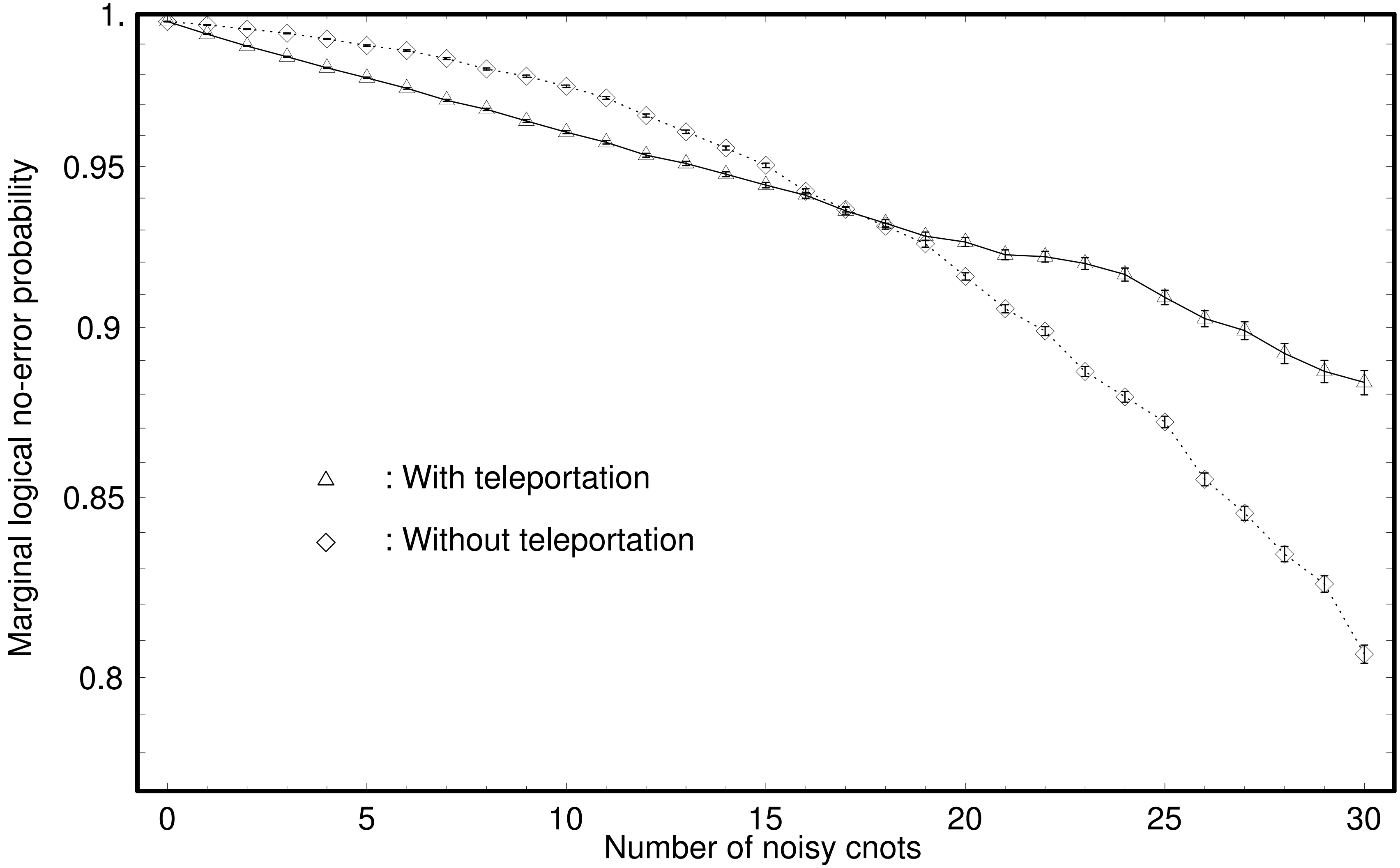


Selective Hadamard error, conditional on success.



Compounding Errors in Sequential Operations

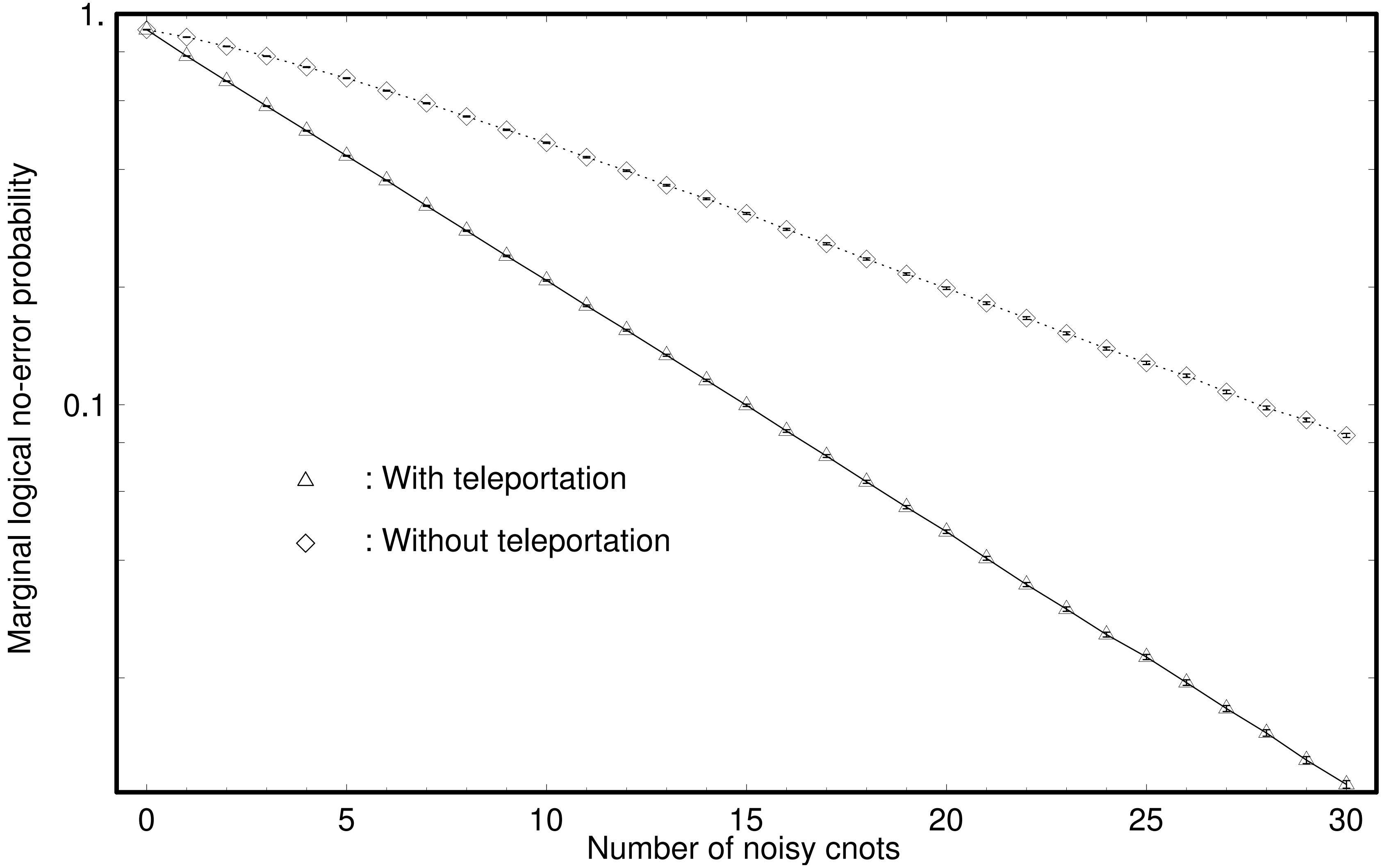
Conditional logical 2-qubit error after x cnots with error correction at level 2.



The probability of a physical cnot error is $1\,\%$.

Success Probability in Sequential Operations

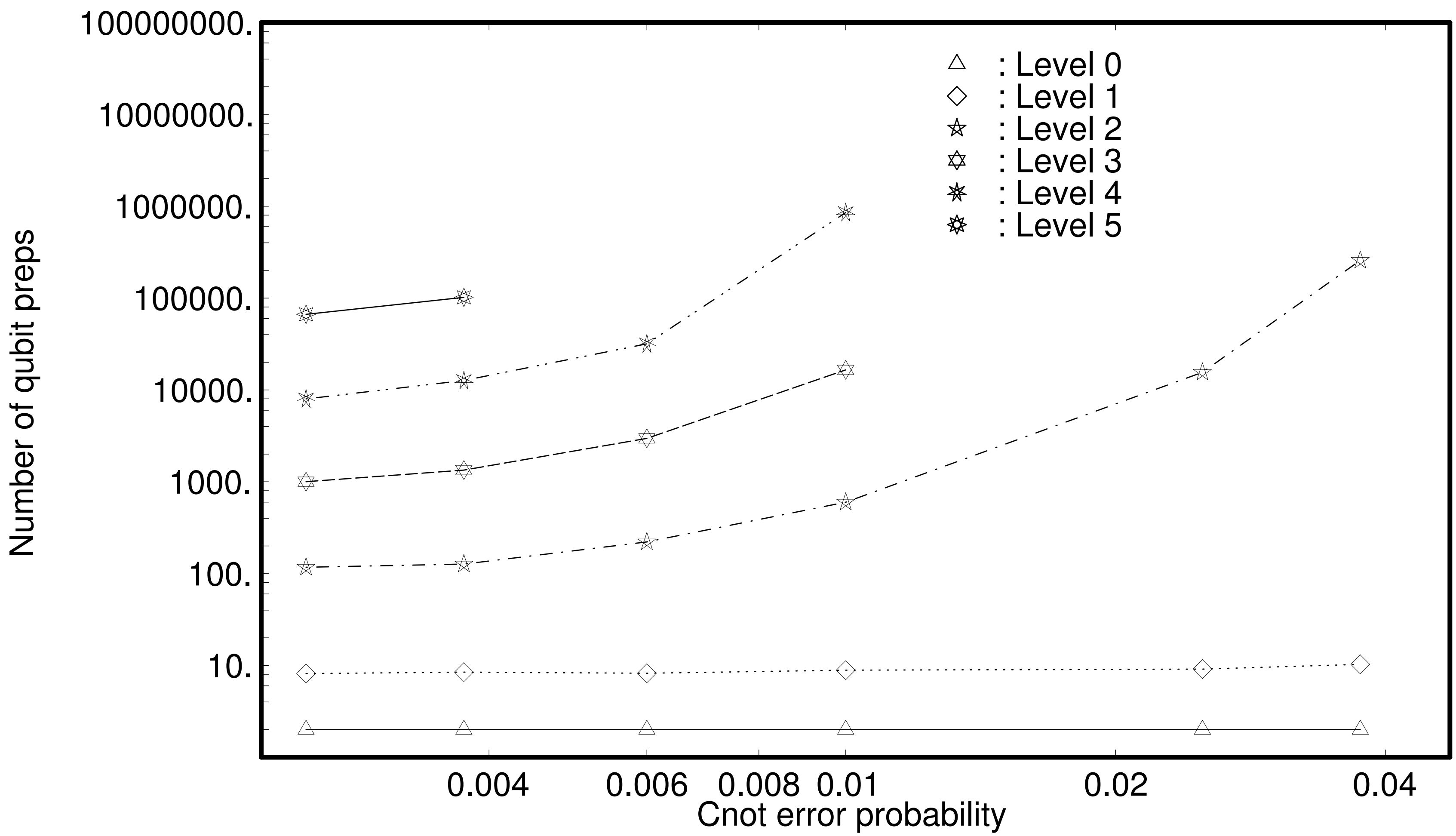
Probability of no detected error after x cnots with error correction at level 2.



The probability of a physical cnot error is $1\,\%$.

Resource Overheads per Logical Qubit

qubits prepared for a log. Bell state at various levels, with error correction.



Fault-Tolerant Simulation Benchmarks

- Benchmarking an architecture.

  Given:
  - A Clifford-Pauli fault-tolerant scheme.
  - Error model.

  Goals:
  - Show that the logical error model is ok.
  - Establish logical error rates.
  - Determine resource requirements.

  If possible, obtain Resources(logical error rates, physical error rates).

- Benchmarking an algorithm

  Given:
  - A Clifford-Pauli fault-tolerant scheme.
  - Error model.
  - An algorithm.

  Goal:
  - Algorithm complexity and success probability.

  . . . simulations can provide probabilistic proofs.

- Issues:
  - Unexpected error-propagation in optimized schemes.
  - Logical errors are typically far from independent.
  - Pseudorandom number generators $\rightarrow$ not a foolproof prob. proof.

Conclusion

- There is evidence that:
  F.-t. QC is possible in principle at error rates well above $1\,\%$.

- But:
  At what error rates is it “practical” to quantum compute with, for example, 10^{10} logical gates and 10^4 logical qubits using technology X?

On Clifford-Pauli Simulators

- Motivation: Benchmark stab.-based f.-t. architectures.

- Some existing Clifford-Pauli simulators:
  - Chung&*al* (2003), for “practical” architectures. (Matlab and Python)
  - Reichardt (2004) [8], exploring the concatenated 7-qubit code with error-detection methods.
  - Aaronson&Gottesman [22], well optimized, some capabilities beyond the Clifford-Pauli group, no Gaussian elimation to achieve quadratic overhead and publicly available. (C)
  - Knill (2004), general purpose, Graph-code normal form to achieve quadratic overhead, fast statistics, needs to be rebuilt. (Octave)

- Capabilities: A few thousand qubits at seconds/operation.

- Bottlenecks:
  - Getting statistics to estimate low logical error-rates.
  - Computer memory.
  - Without taking advantage of sparseness: Significant slowdown.

Contents

References

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