Long-range quantum entanglement in noisy cluster states

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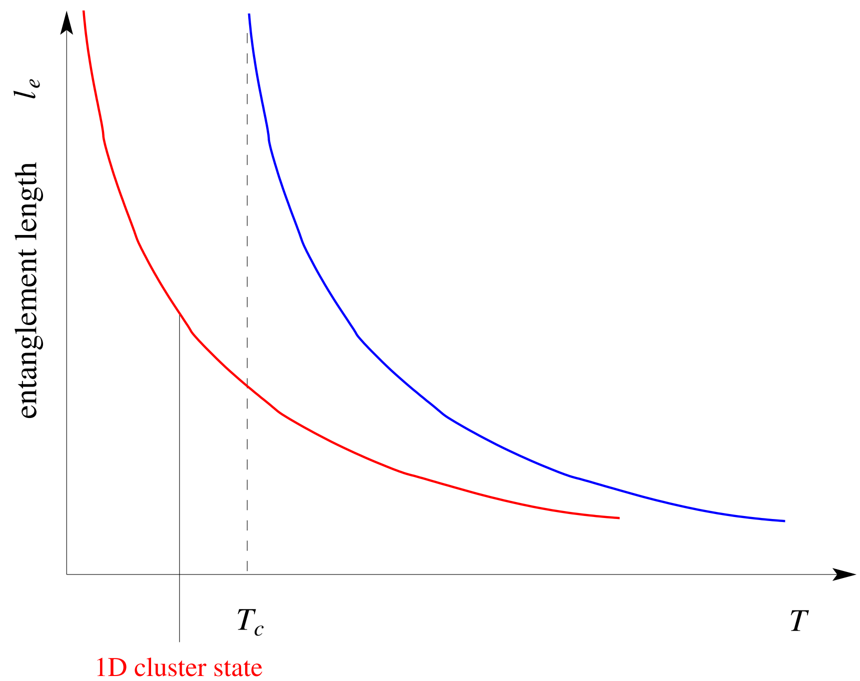
1. Introduction

Can there exist infinite-range entanglement in thermal states at finite temperature?



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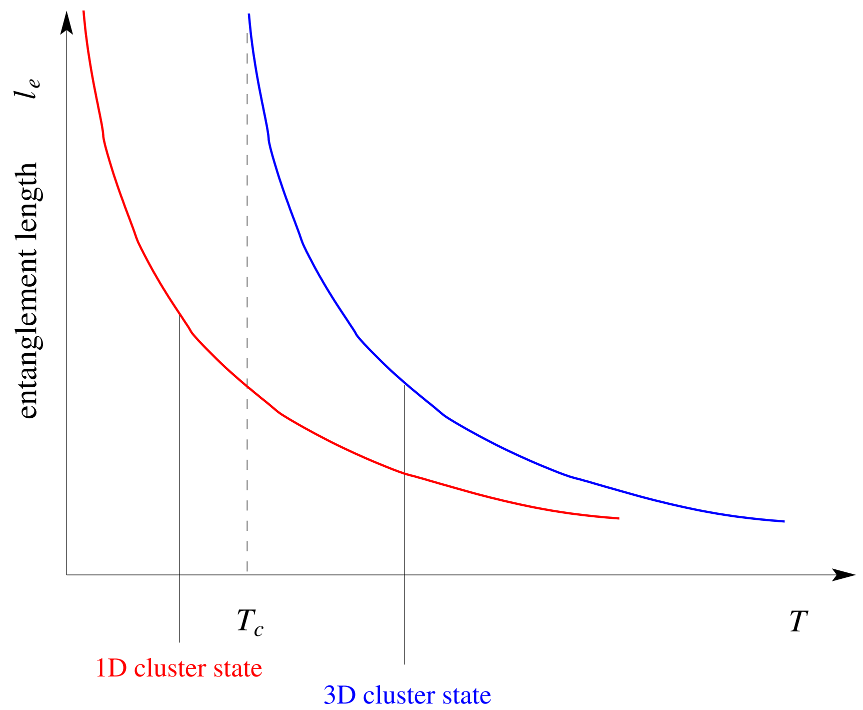

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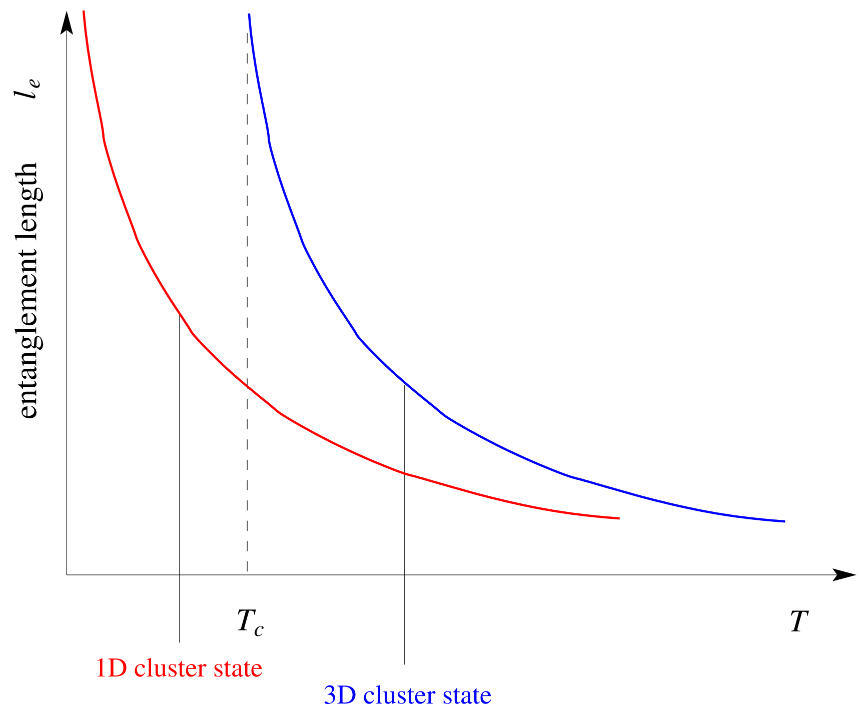
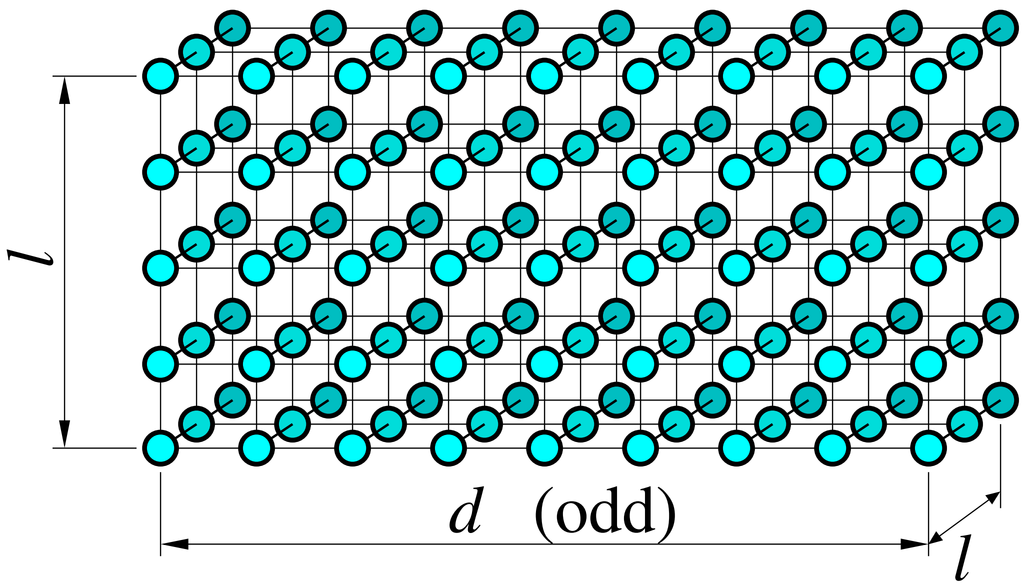


The 3D cluster state provides intrinsic error correction.

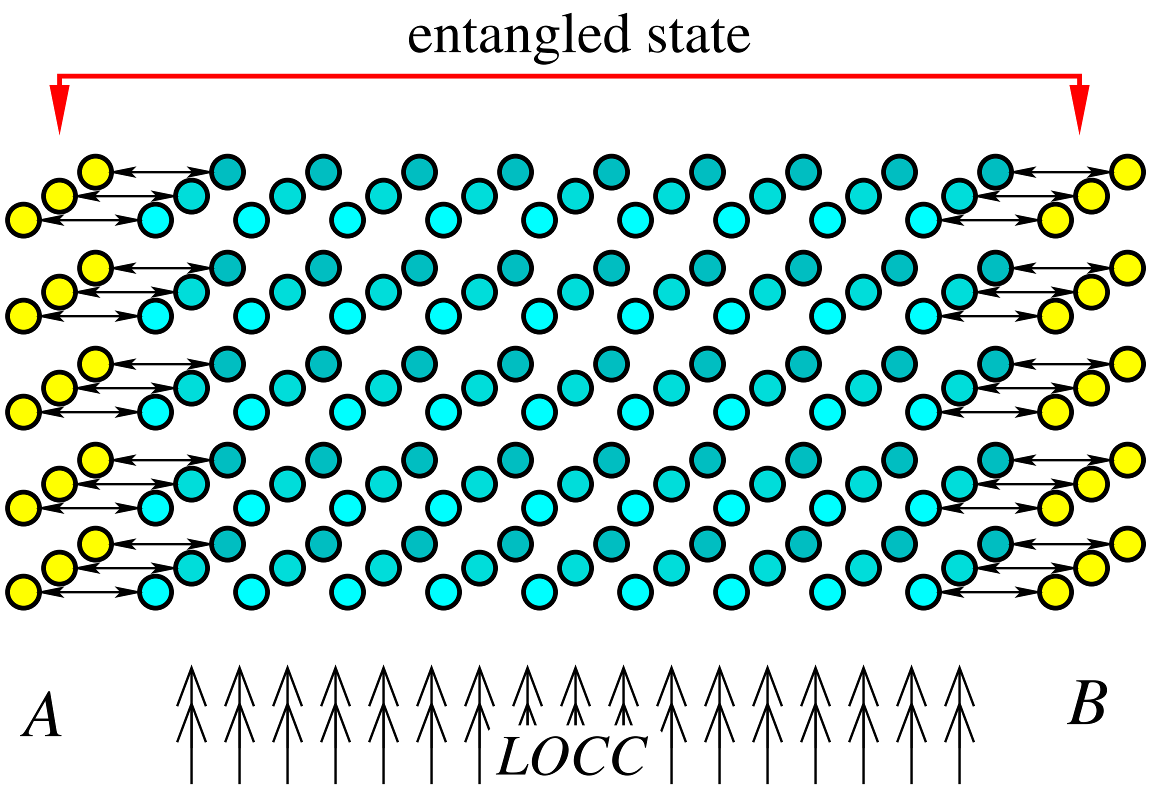
2. Model



$$\rho_{\mathsf{in}}(T) = \frac{1}{Z(T)}\, e^{-H/T},$$

3D thermal cluster state at temperature T.

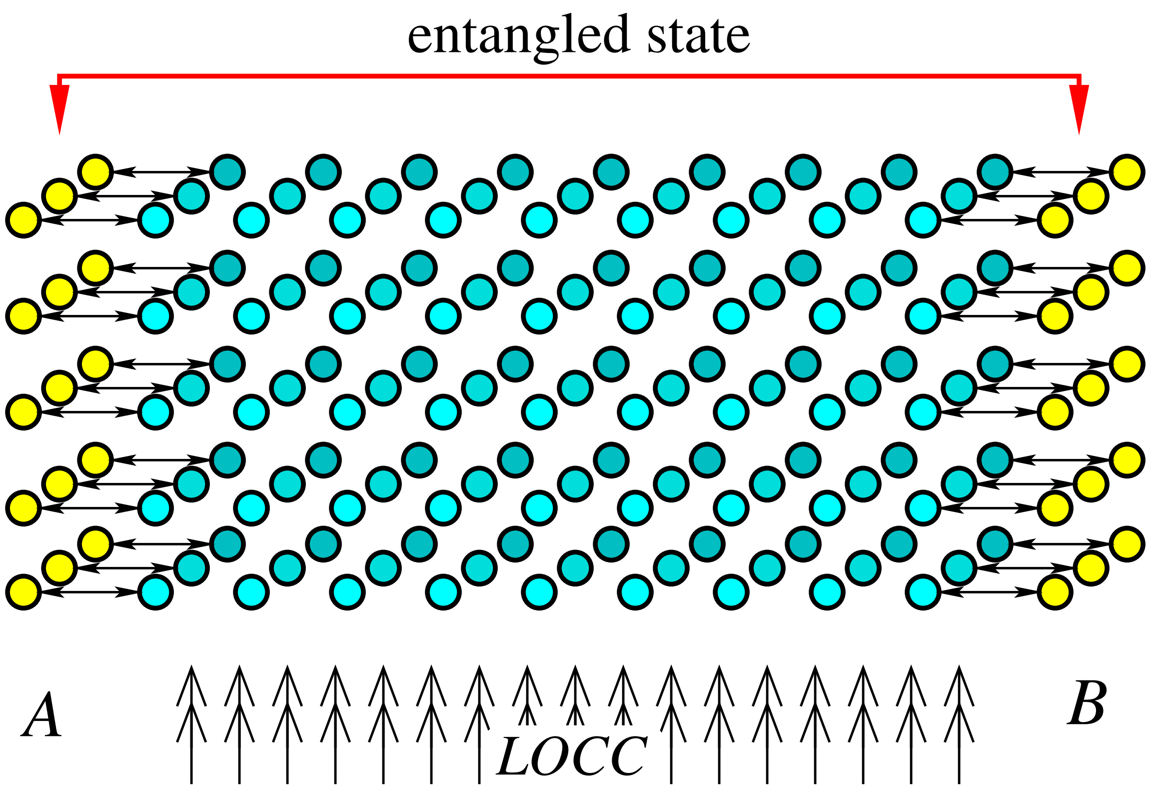
2. Model



$$|\psi_{\text{out}}\rangle_{AB} \cong \overline{|0\rangle}_A \overline{|0\rangle}_B + \overline{|1\rangle}_A \overline{|1\rangle}_B$$

Output: encoded Bell state among A and B.

2. Model



Interested in the *localizable entanglement*.

Entanglement length $L_e = d$.

2. Model: 3D thermal cluster state

We consider the Hamiltonian

$$H = -\frac{\Delta}{2} \sum_{a \in \mathcal{C}} K_a, \qquad (1)$$

where $K_a = X_a \bigotimes_{b \in \text{nbgh}(a)} Z_b$.

The thermal cluster state $\rho_{\mathcal{C}}(T) = \frac{1}{Z} e^{-\beta H}$, $\beta \equiv 1/T$, then is

$$\rho_{\mathcal{C}}(T) = \frac{1}{2^{|\mathcal{C}|}} \prod_{a \in \mathcal{C}} \left(I + \tanh\left(\frac{\beta\Delta}{2}\right) K_a \right). \qquad (2)$$

- $\rho_{\mathcal{C}}(T = 0)$ is a cluster state $|\phi\rangle_{\mathcal{C}}\langle\phi|$.

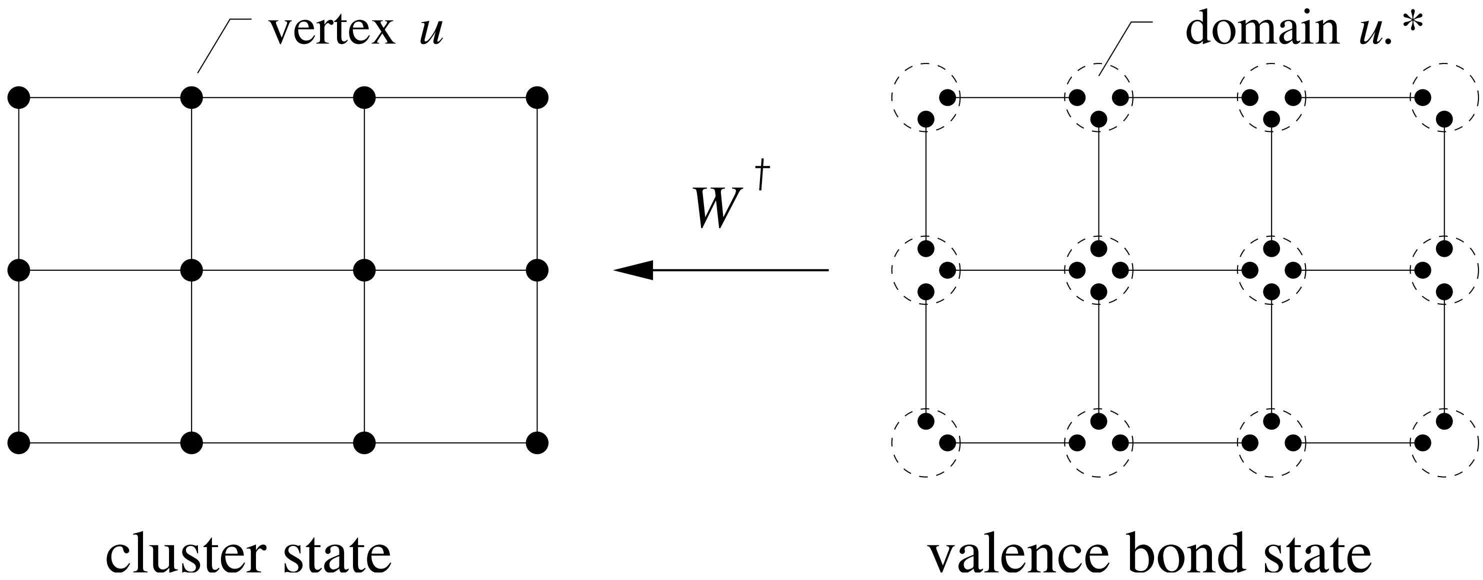
3. Result

For the 3D thermal cluster state $\rho(T)$ a transition between infinite and finite entanglement length occurs between

$$0.30\,\Delta \leq T_c \leq 1.13\,\Delta. \tag{3}$$

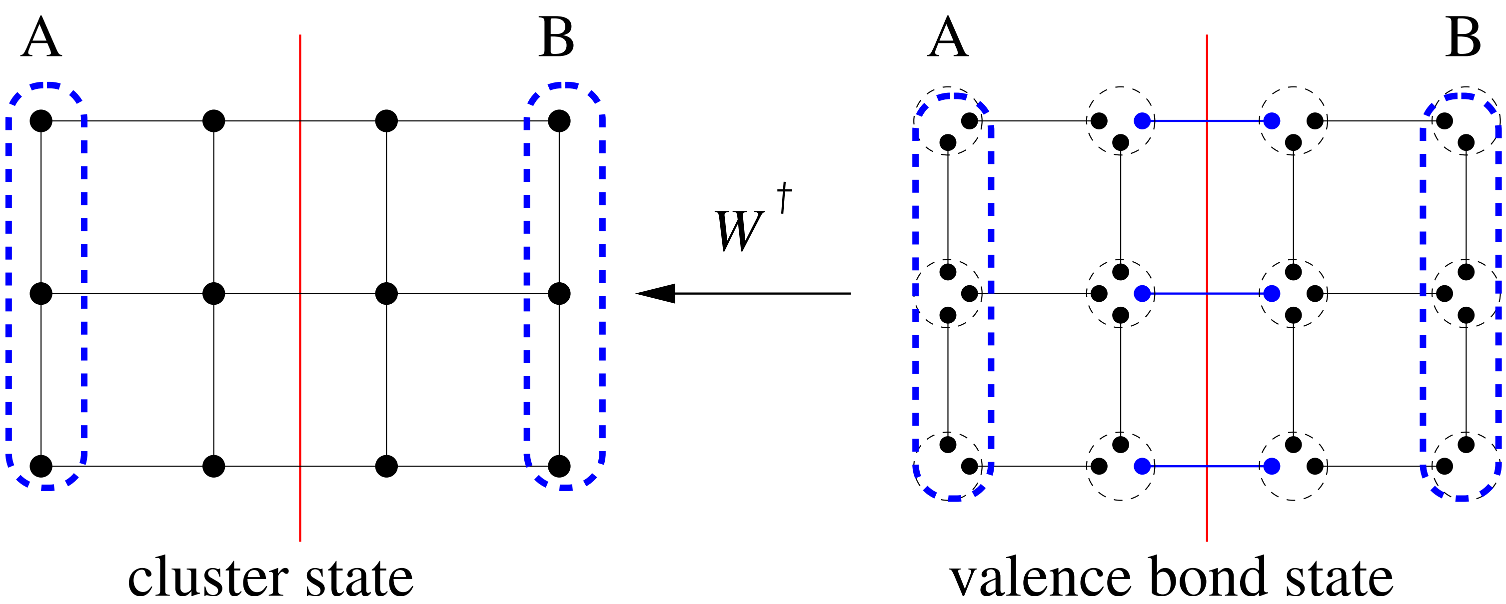
Δ: the energy gap of H.

4. Upper bound to T_c



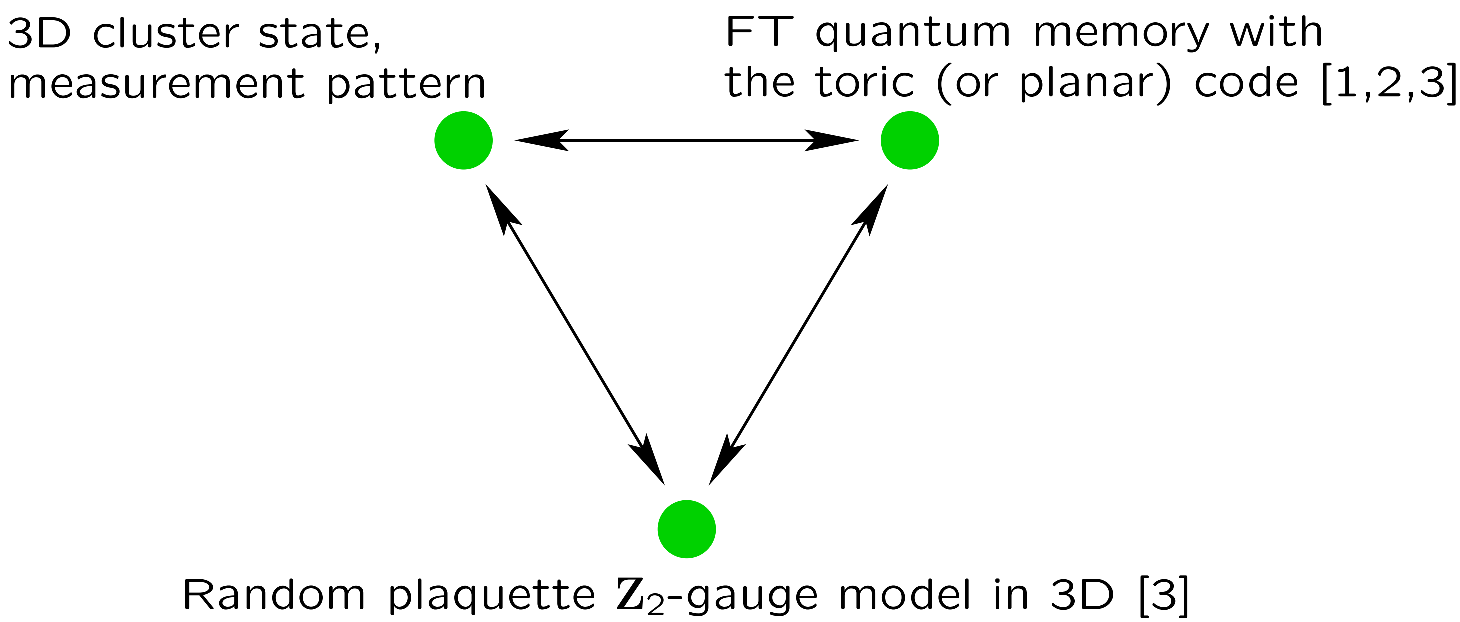
$$W^\dagger = \bigotimes_{u \in \mathcal{C}} |0\rangle_u \langle 0^{|u.*|}| + |1\rangle_u \langle 1^{|u.*|}|. \tag{4}$$

4. Upper bound to T_c



Applying the PPT separability criterion to the Bell pairs in the VBS state across the cut yields $T_c \leq 1.13\,\Delta$.

5. Lower bound to T_c



[1] A. Kitaev, quant-ph/9707021 (1997).

[2] S. Bravyi, A. Kitaev, quant-ph/9810092 (1998).

[3] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, quant-ph/0110143 (2001).

5.1 Error model

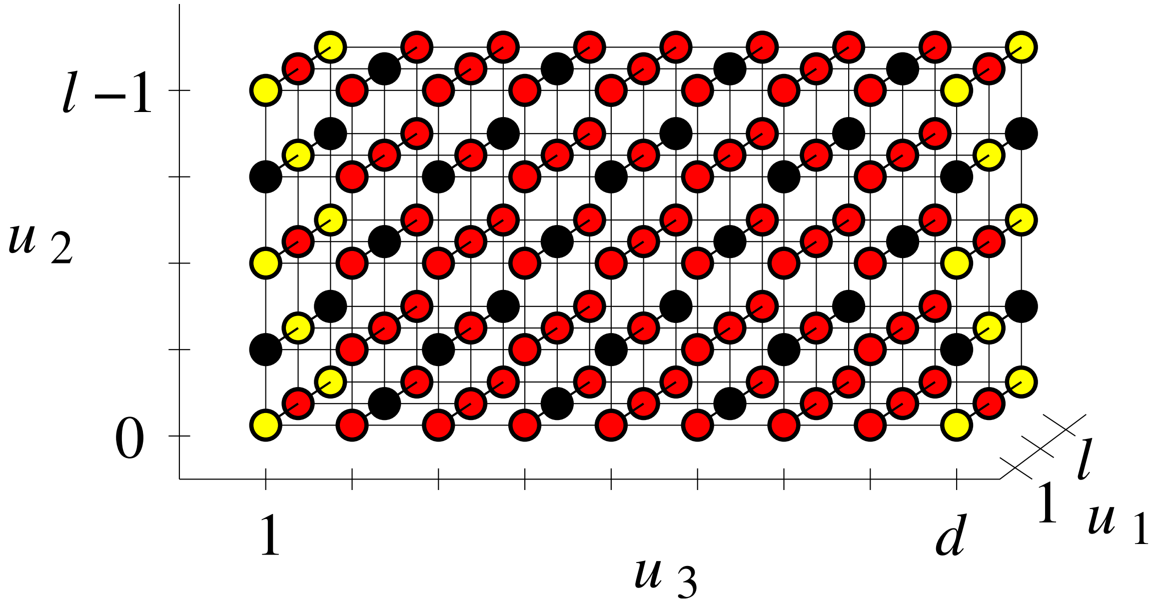
The thermal cluster state

$$\rho_{\mathcal{C}}(T) \;=\; \frac{1}{2^{|\mathcal{C}|}} \prod_{a \in \mathcal{C}} \left(I + \tanh\left(\frac{\beta\Delta}{2}\right) K_a \right)$$

is equivalent to local phase errors Z_a applied to the perfect cluster state, with probability

$$p = \frac{1}{1+\exp(\beta\Delta)}. \tag{5}$$

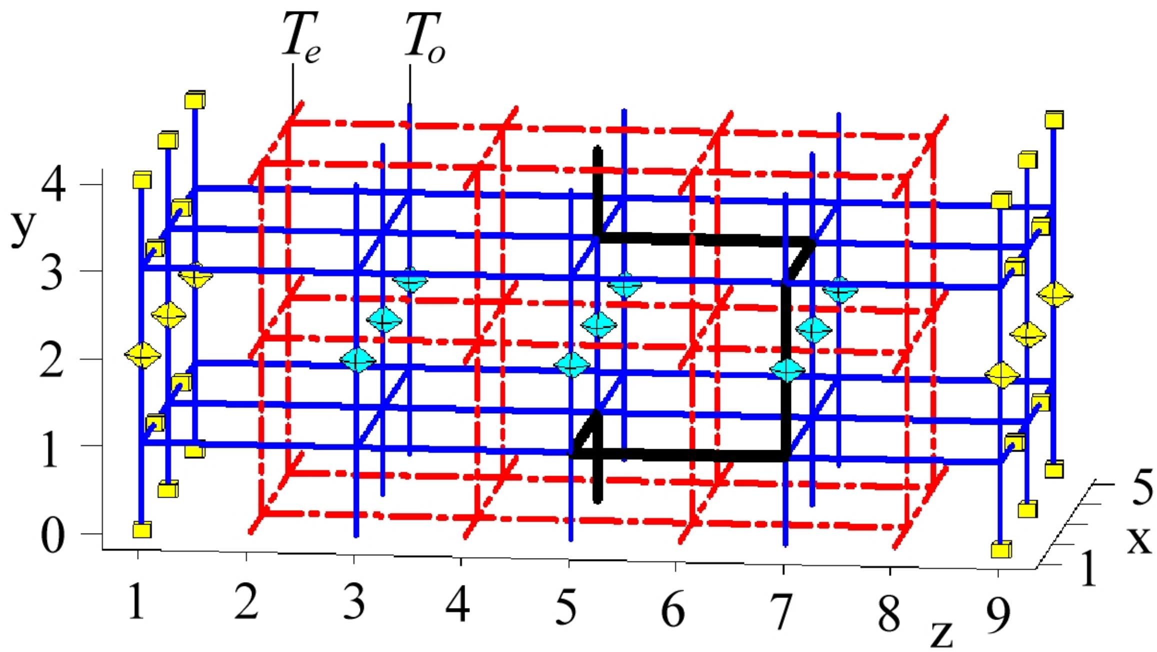
5.2 The measurement pattern



$$\begin{array}{lll} \text{: Not measured.} & u_3 = 1, d,\ u_1 + u_2 = \text{odd}, & \\ \text{: } \sigma_z\text{-measurement.} & u = (e[\text{ven}], e, e), (o[\text{dd}], o, o), & (6) \\ \text{: } \sigma_x\text{-measurement.} & \text{otherwise.} & \end{array}$$

If the qubits $\bigcirc_L$ (left) are entangled with $\bigcirc_R$ (right) after the local measurements, they must have been entangled before.

5.3 Mapping to the Z_2 gauge model



What needs to be shown:

1. Without errors: post meas. $|\psi\rangle_{LR}$ is an encoded Bell state
2. Considering errors:
   - Lattices for $\mathbf{Z}_2$ gauge model: T_o and T_e (simple cubic, double spacing). T_e [T_o] mediates $\overline{ZZ}$- [$\overline{XX}$]- correlations.
   - Elementary errors on edges and syndrome bits on vertices of T_e, T_o.
   - Harmful errors: homologically nontrivial error cycles.

5.4 Lower bound to T_c

- For the described measurement pattern the measurement outcomes are *dependent* $\longrightarrow$ error detection and correction.
- A *random plaquette Z_2-gauge model in 3D* [1] describes the performance of error correction.
  - high-temp disordered phase: error correction fails
  - low-temp ordered phase: error correction successful

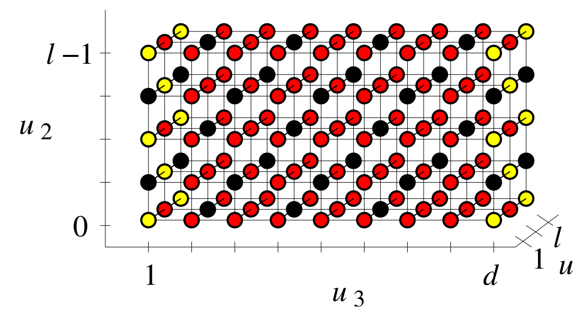
Critical error probability [2], temperature:

$$p_c = 0.033 \iff T_c = 0.3\,\Delta.$$

[1] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, quant-ph/0110143 (2001).
[2] T. Ohno, G. Arakawa, I. Ichinose and T. Matsui, quant-ph/0401101 (2004).

5.5 Alternative explanation for the MP

$\bigcirc$:	Not measured.	$u_3 = 1, d,\ u_1 + u_2 = \text{odd}$,
$\bullet$:	σ_z-measurement.	$u = (e[\text{ven}], e, e), (o[\text{dd}], o, o)$,
$\bullet$:	σ_x-measurement.	otherwise.



May artificially split the measurement pattern into two steps:

1. Measure the qubits with $u_1 + u_2 = \text{even}$, $\forall u_3$.

   $\Longrightarrow$ 1D cluster state encoded with the planar code.

2. Measure the remaining qubits, except $\bigcirc_L$ and $\bigcirc_R$ (all X).

   $\cong$ fault-tolerant encoded $\overline{X}$ measurements at $2 \leq u_3 \leq d-1$

   $\Longrightarrow$ encoded Bell state between $\bigcirc_L$ and $\bigcirc_R$.

6. More general errors

- X,Y-errors can also be corrected.

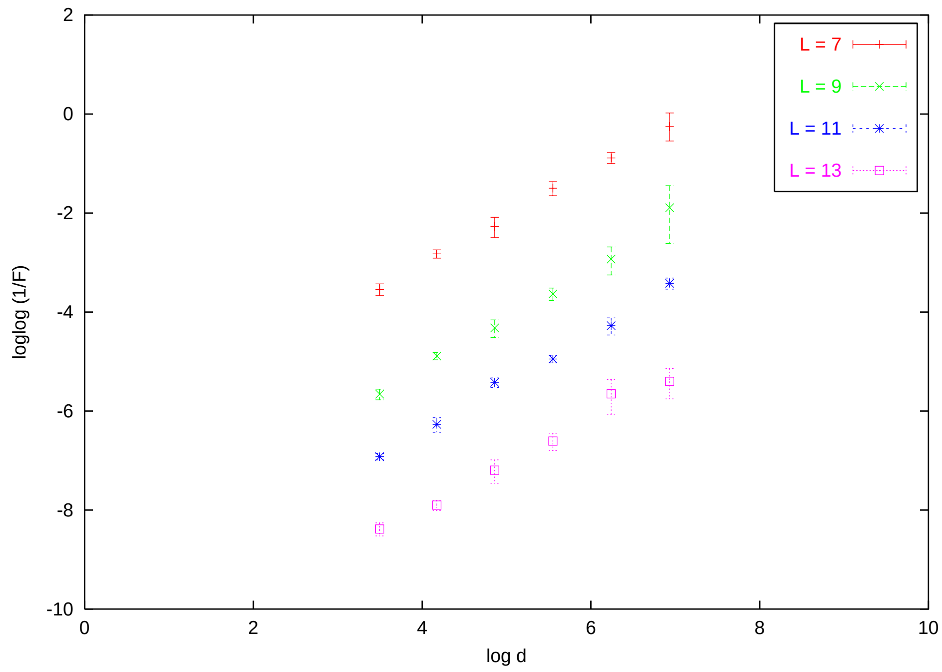
Local depolarizing channel with error prob. $p_x = p_y = p_z = \frac{p'}{3}$:

- Error threshold $p_c' = 1.4\%$.
- Note: measurement pattern contains σ_z-measurements. If corresponding cluster qubits left out from the beginning, then $p_c'' = 3/2\, p_c = 4.9\%$.

7. Finite size effects

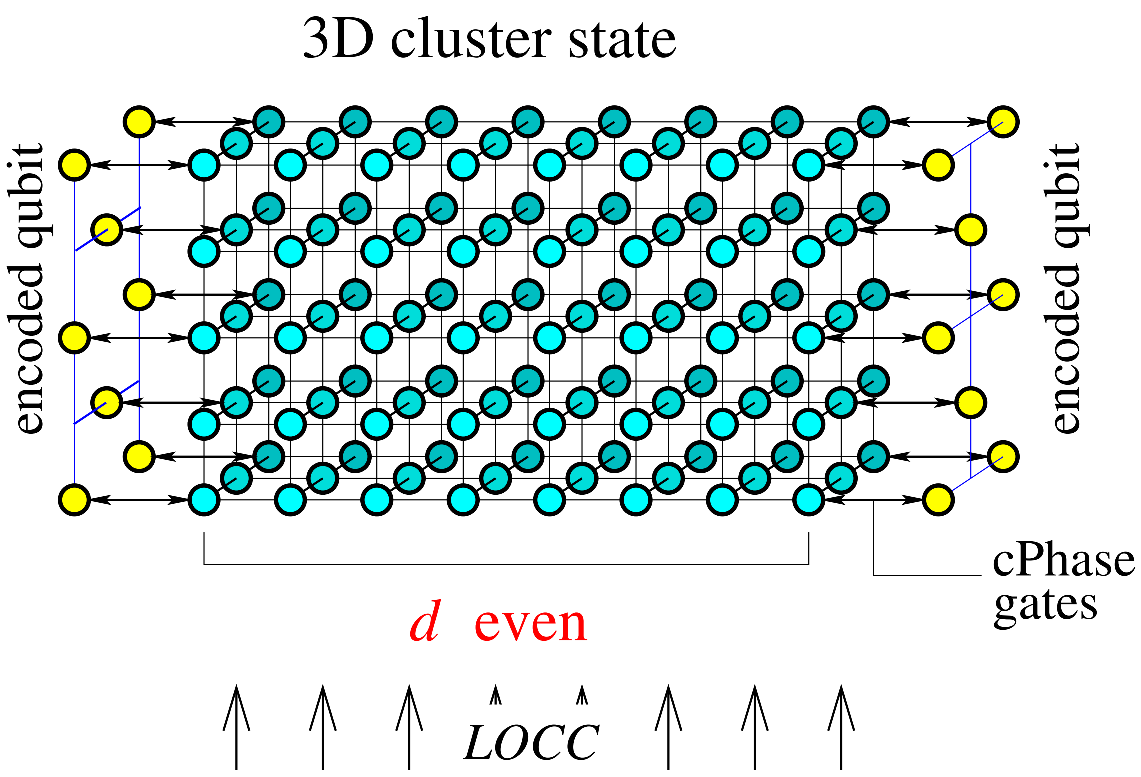
Numerically: For the Bell state fidelity F we find

$$F \sim \exp\left(-d\, k_1(p) \exp(-k_2(p)\, l)\right). \tag{7}$$



Consequence: for constant F, code block length $l \sim \log(d)$.

8. An application



Fault-tolerant encoded long-distance cPhase gate, mediated via short-range interaction and LOCC.

9. Sumary

- The thermal cluster state exhibits a transition from infinite to finite entanglement length at a nonzero temperature T_c,

$$0.3\,\Delta \leq T_c \leq 1.13\,\Delta.$$

  (Δ: energy gap of the Hamiltonian)

- The reason for this behavior is an intrinsic error correction capability of 3D cluster states.

- Have established a connection

$$\text{cluster states} \Longleftrightarrow \text{surface codes}.$$