Optimal Unravellings for Feedback Control in Linear Quantum Systems

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Quantum Feedback Control in Linear Systems

Quantum feedback is rapidly developing, especially experimentally:

- Freezing a conditional state in cavity QED (Orozco & co, PRL, 2002)
- Sub-SQL adaptive phase estimation (Mabuchi & co, PRL, 2002)
- Deterministic spin squeezing (Mabuchi & co, Science, 2004).

For the latter two cases the theory (Wiseman, PRL, 1995; Thomsen, Mancini & Wiseman, PRA(RC), 2002) involved *linearizing the phase-space dynamics* of the measured systems.

For such linear(izable) systems, *classical* feedback control theory can be applied to good effect (Belavkin, 1987; Doherty & Jacobs & co., PRA, 1999, 2000).

New Results for Linear Quantum Systems

- Quantum fluctuation-dissipation relation
- General theory of conditional dynamics (*unravellings*)
- Corresponding Heisenberg picture equations
- N&S conditions for the existence of a SQL
- N&S condition for V to be the solution for the conditioned covariance matrix under *some* unravelling.
- N&S condition for this V to be a *stabilizing* solution
- Semi-Definite Program for a class of feedback control problems.

Outline

1. Quantum master equations and their unravellings

2. Quantum systems in phase-space

3. Linear quantum dynamics in phase-space

4. Optimal quantum control

5. A worked example

1. Quantum master equations

The QME is the most general autonomous differential equation for the state ρ of a quantum system (Lindblad, 1976):

$$\hbar\dot{\rho} = -i[\hat{H}, \rho] + \sum_{l=1}^{L} \mathcal{D}[\hat{c}_l]\rho \equiv \mathcal{L}_0\rho \qquad (1)$$

- $\hat{H} = \hat{H}^\dagger$ is the system Hamiltonian
- $\hat{\mathbf{c}} = (\hat{c}_1, \cdots, \hat{c}_L)^\top$ is a vector of bounded operators
- $\mathcal{D}[\hat{c}]\rho \equiv \hat{c}\rho\hat{c}^\dagger - \left(\hat{c}^\dagger\hat{c}\rho + \rho\hat{c}^\dagger\hat{c}\right)/2.$

Widely used in atomic, optical, and nuclear physics.

Unravelling quantum master equations

A QME typically applies if the system is coupled weakly to a large bath, and the bath is ignored (traced over). Because the system and bath entangle, ρ becomes mixed.

But it is not always appropriate to ignore the bath — often it can be measured, yielding information about the system and producing a conditioned system state ρ_c more pure than ρ.

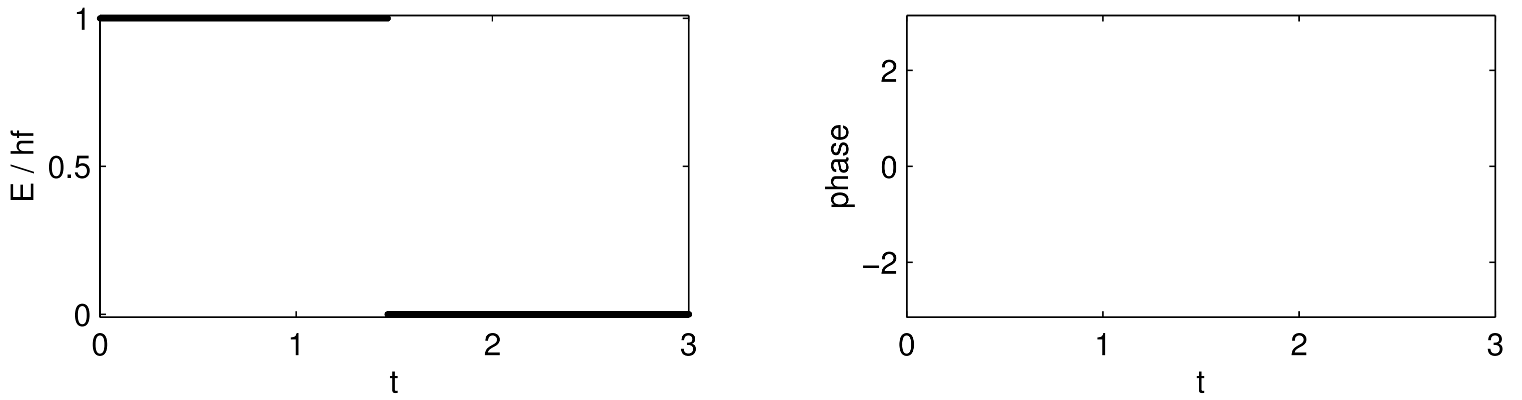
If a QME can be derived then the bath can be measured repeatedly, much faster than any relevant system rate *without invalidating the QME*. We say the stochastic evolution for $\rho_c(t)$ *unravels* the QME:

$$\mathrm{E}[\rho_c(t)] = \rho(t) = \exp[\mathcal{L}_0 t/\hbar]\rho(0). \quad (2)$$

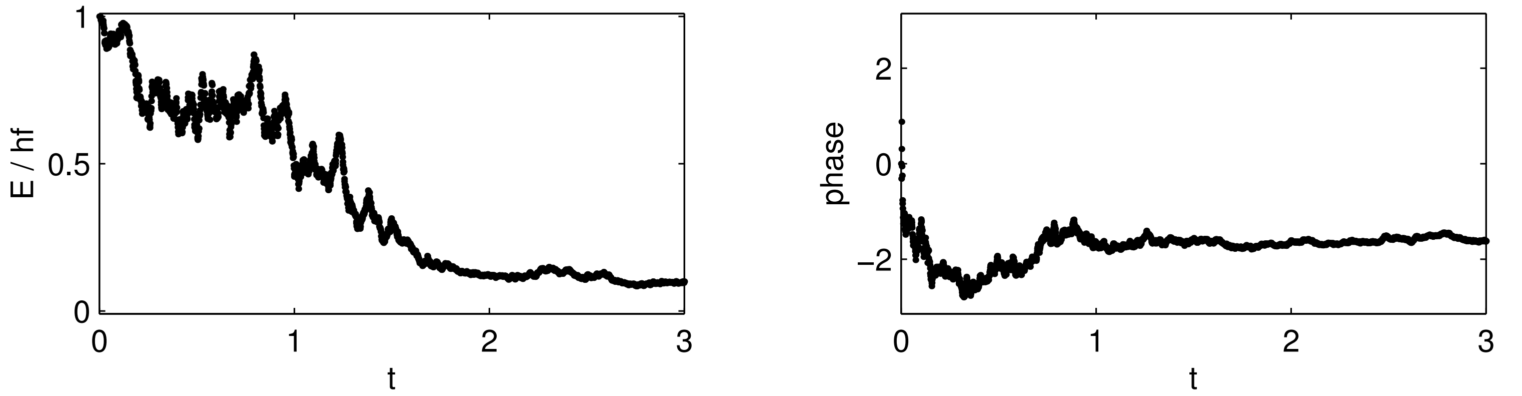
Different ways of measuring the bath lead to different unravellings.

Example: Decay of an Excited State Atom — Unravelled Evolution (Quantum Trajectories)

Direct Detection (Avalanche Photodiode):



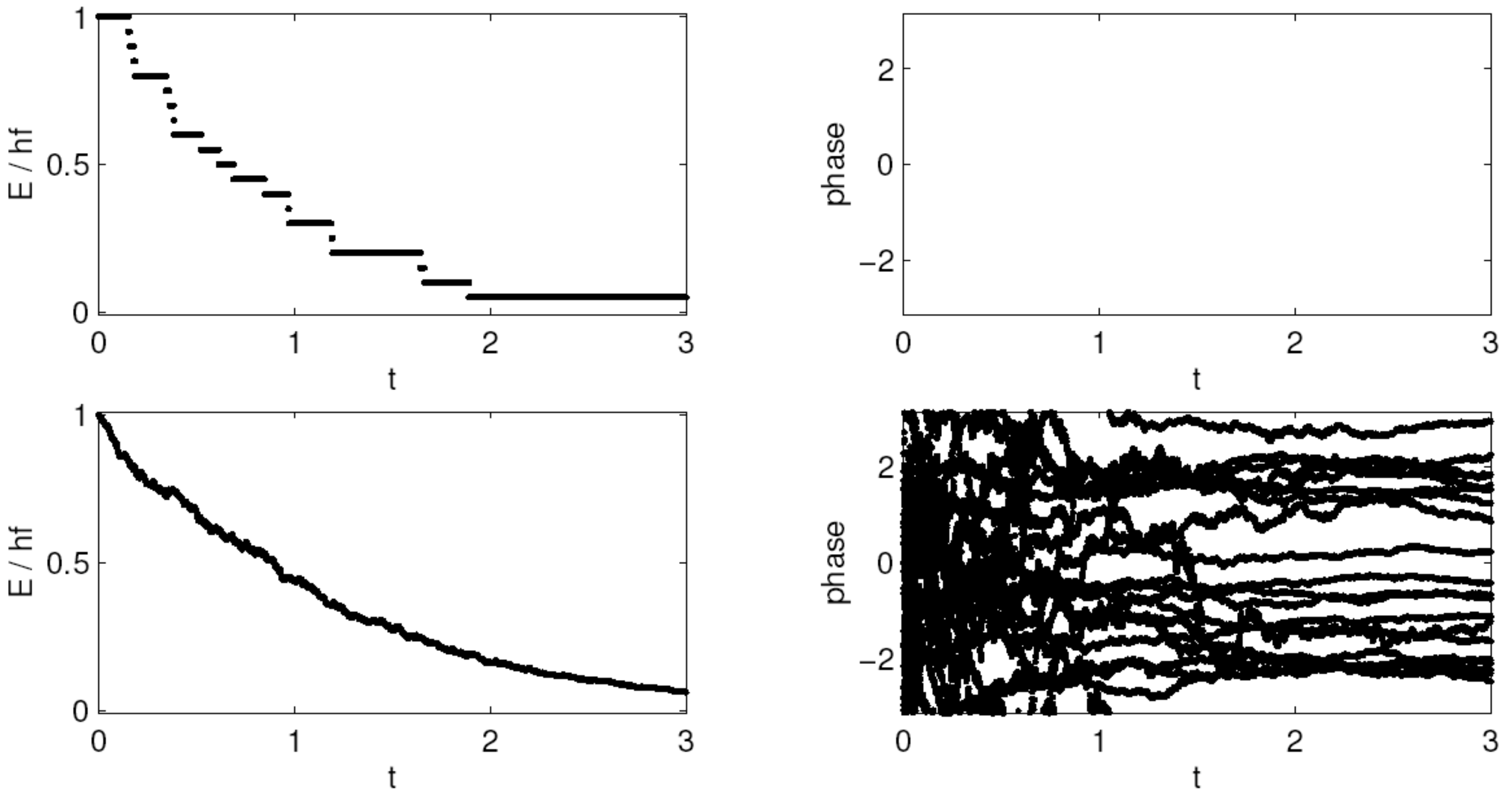
Heterodyne Detection (Laser and Photoreceiver):



Example: Decay of an Excited State Atom — Ensemble Average Evolution

Direct Detection (Avalanche Photodiode):

Heterodyne Detection (Laser and Photoreceiver):



2. Quantum systems in Phase Space

Consider a systems of N degrees of freedoms, each with a canonically conjugate pair: $[\hat{q}_n, \hat{p}_m] = i\hbar\delta_{nm}$. Let

$$\hat{\mathbf{x}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\top . \tag{3}$$

Then $[\hat{x}_n, \hat{x}_m] = i\hbar\Sigma_{nm}$ where Σ is a $(2N) \times (2N)$ symplectic matrix:

$$\Sigma = \bigoplus_{n=1}^{N} \sigma_n, \quad \text{where} \quad \sigma_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

We define the mean $\langle \hat{\mathbf{x}} \rangle = \mathrm{Tr}[\rho \hat{\mathbf{x}}]$ and fluctuation $\Delta\hat{\mathbf{x}} = \hat{\mathbf{x}} - \langle \hat{\mathbf{x}} \rangle$.

The Covariance Matrix and Gaussian States

The covariance matrix is defined by

$$V_{nm} = \left(\langle \Delta \hat{x}_n \Delta \hat{x}_m \rangle + \langle \Delta \hat{x}_m \Delta \hat{x}_n \rangle \right) /2. \tag{5}$$

the identity $V_{nm} + i\hbar \Sigma_{nm}/2 = \mathrm{Tr}\left[\rho \Delta \hat{x}_n \Delta \hat{x}_m\right]$ and positivity of ρ make it *necessary* that V satisfies the LMI

$$V + i\hbar \Sigma/2 \geq 0. \tag{6}$$

This LMI is a generalization of the Heisenberg uncertainty relation.

Gaussian quantum states are states with a Gaussian Wigner function with mean vector $\langle \hat{\mathbf{x}} \rangle$ and covariance matrix V. For such states it is also *sufficient* that V satisfy Eq. (6).

3. Linear Dynamics

A *linear* system is one for which $\hat{H}$ is quadratic, and $\hat{\mathbf{c}}$ linear, in $\hat{\mathbf{x}}$:

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^\top G\hat{\mathbf{x}} - \hat{\mathbf{x}}^\top \Sigma B\mathbf{u}(t), \quad \begin{pmatrix} \mathrm{Re}[\hat{\mathbf{c}}] \\ \mathrm{Im}[\hat{\mathbf{c}}] \end{pmatrix} = \bar{C}\hat{\mathbf{x}}, \tag{7}$$

where G is real and symmetric and B and $\bar{C}$ are real.

The QME then has a Gaussian state as its solution, with

$$d\langle\hat{\mathbf{x}}\rangle/dt = A\langle\hat{\mathbf{x}}\rangle + B\mathbf{u}(t) \tag{8}$$

$$dV/dt = AV + VA^\top + D. \tag{9}$$

The diffusion and drift matrices are $D = \bar{h}\Sigma\bar{C}^\top\bar{C}\Sigma^\top$, $A = \Sigma G + \Sigma\bar{C}^\top S\bar{C}$, where $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Conditional Evolution for Linear Systems

If we require the output (measured bath observables) to be linear in $\hat{\mathbf{x}}$ then the most general output compatible with the QME is

$$\mathbf{y} = C\langle\hat{\mathbf{x}}\rangle_{\rm c} + \frac{d\mathbf{w}}{dt}. \tag{10}$$

- $C = 2\sqrt{U/\bar{h}}\bar{C}$, where unravelling matrix $U = \frac{1}{2}\begin{pmatrix} \mathrm{H}+\mathrm{Re}[\Upsilon] & \mathrm{Im}[\Upsilon] \\ \mathrm{Im}[\Upsilon] & \mathrm{H}-\mathrm{Re}[\Upsilon] \end{pmatrix}$.
- i.e. infinitely many different unravellings U constrained only by: (i) $\Upsilon = \Upsilon^\top$, (ii) $\mathrm{H} = \mathrm{diag}(\eta_1, \cdots, \eta_K)$ with $0 \leq \eta_k \leq 1$, and (iii) $U \geq 0$.
- $d\mathbf{w}$ is a vector of Wiener increments: $d\mathbf{w}d\mathbf{w}^\top = I dt$.

For linear systems, the state conditioned on $\mathbf{y}(t)$ is Gaussian.

Quantum Kalman Filter Equations

$d\rho_c(t)$ can be expressed by the conditional moment equations:

$$d\langle\hat{\mathbf{x}}\rangle_{\rm c} = [A\langle\hat{\mathbf{x}}\rangle_{\rm c} + B\mathbf{u}(t)]\,dt + \left(V_{\rm c}C^\top + \Gamma^\top\right)d\mathbf{w} \tag{11}$$

$$\dot{V}_{\rm c} = AV_{\rm c} + V_{\rm c}A^\top + D - (V_{\rm c}C^\top + \Gamma^\top)(CV_{\rm c} + \Gamma), \tag{12}$$

Here $\Gamma = -\sqrt{\hbar U}\,S\bar{C}\Sigma^\top$ and (as before) $C = 2\sqrt{U/\hbar}\bar{C}$.

Note that Eq. (12) is deterministic! The final term causes a reduction in uncertainty (i.e. in the eigenvalues of $V_{\rm c}$).

Remarkably, the set of possible $V_{\rm c}^{\rm ss}$, for all possible unravellings U, is simply the solution set $\{V\}$ satisfying

$$D + AV + VA^\top \geq 0 \text{ and } V + i\hbar\Sigma/2 \geq 0. \tag{13}$$

4. Optimal Quantum Feedback Control

In feedback control, the optimal solution to a well-defined problem is always

$$\hat{H}_{\rm fb}(t) = \hat{f}(\rho_{\rm c}(t), t) \tag{14}$$

That is, $\mathbf{y}(s)$ for $s < t$ is irrelevant except in so far as it determines $\rho_{\rm c}(t)$, as this is the observer's *state of knowledge*.

For LQG control (**L**inear dynamics, **Q**uadratic cost function, **G**aussian noise), the optimal solution is

$$\hat{H}_{\rm fb}(t) = \hat{\mathbf{x}}^\top \Sigma B \mathbf{u}(t), \text{ with } \mathbf{u}(t) = -K(t)\langle\hat{\mathbf{x}}\rangle_{\rm c}(t), \tag{15}$$

where the matrix $K(t)$ can be determined from A, B, and the cost functions, independently of D, C, and Γ.

Manipulability and Asymptotic Terminal-only Cost

- **Manipulable** system: B is full-rank
  (i.e. arbitrary displacements in phase-space can be performed)

- **Asymptotic** terminal-only cost function:

$$\Lambda = \mathrm{Tr}\left[\hat{\mathbf{x}}^\top P_1 \hat{\mathbf{x}} \rho_{\mathrm{ss}}\right] = \mathrm{E}_{\mathrm{ss}}\left\{\mathrm{Tr}\left[\hat{\mathbf{x}}^\top P_1 \hat{\mathbf{x}} \rho_{\mathrm{c}}\right]\right\}$$

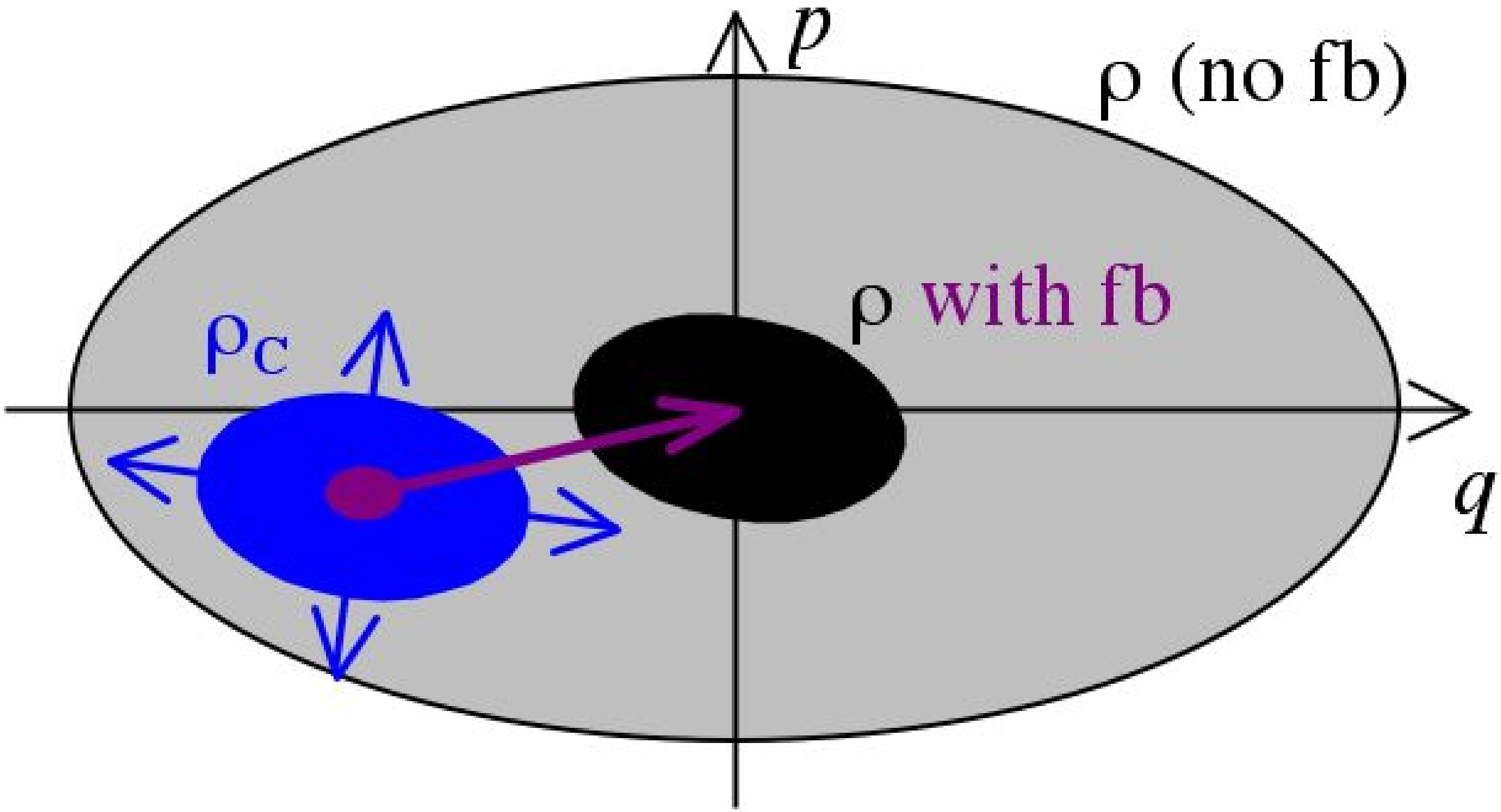
  where P_1 is a PSD symmetric real matrix.

For any problem with terminal-only costs, the optimal K is unbounded. With manipulability we can choose K such that in

$$d\langle\hat{\mathbf{x}}\rangle_{\mathrm{c}} = (A - BK)\langle\hat{\mathbf{x}}\rangle_{\mathrm{c}} + \left(V_{\mathrm{c}}C^\top + \Gamma^\top\right) d\mathbf{w}, \qquad (16)$$

the damping will overwhelm the noise so we can set $\langle\hat{\mathbf{x}}\rangle_{\mathrm{c}} = \mathbf{0}$.

Picture of the Feedback



The stationary state with feedback is completely characterized by V_c^{ss}.

Optimizing the Unravelling

Note that $\mathrm{Tr}\left[\hat{\mathbf{x}}^\top P_1 \hat{\mathbf{x}} \rho_c\right] = \langle \hat{\mathbf{x}} \rangle_c^\top P_1 \langle \hat{\mathbf{x}} \rangle_c + \mathrm{tr}[P_1 V_c]$.

Hence the minimum cost, when $\langle \hat{\mathbf{x}} \rangle_c = \mathbf{0}$, with $V_c \to V$ as $t \to \infty$, is

$$\Lambda = \mathrm{tr}[P_1 V], \text{ where } D + AV + VA^\top \geq 0 \text{ and } V + i\bar{h}\Sigma/2 \geq 0 \qquad (17)$$

Question:[1] Given the deterministic no-feedback dynamics A, D (or $G, \bar{C}$), what is the *optimal unravelling* U for *minimizing* the cost Λ?

It turns out that the optimal V can be solved *efficiently* (in the system size N) using **Semi-Definite Programming**. Finding a suitable unravelling U given this V is also a SDP.

[1]Note that classically this is *meaningless*, as A, D do not place any constraints on how the system can be measured, because classically there is no back-action noise.

5. Example

Consider the system described by (setting $\hbar = 1$)

$$\dot{\rho} = -i[(\hat{q}\hat{p} + \hat{p}\hat{q})/2, \rho] + \mathcal{D}[\hat{q} + i\hat{p}]\rho, \tag{18}$$

where the output arising from the second term may be monitored. Equivalently

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}$$

Optical realization: a damped cavity containing an on-threshold parametric down converter with p the squeezed quadrature.

In that case the system could be displaced in its phase space by coherent driving, so we could take it to be manipulable.

The Control Problem

- Say the aim is to produce a stationary state where $q = p$ as nearly as possible. A suitable cost function to be minimized is

$$\Lambda = \left\langle (\hat{q} - \hat{p})^2 \right\rangle_{\rm ss} = \mathrm{tr}\left[P_1 V_{\rm ss}\right] \text{ with } P_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \qquad (20)$$

- Assuming manipulability (so that $V_{\rm ss} = V$) we find

$$\Lambda \approx 1.11769 \text{ for } U = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \qquad (21)$$

where $\theta \approx 0.277896\pi$. Physically, this means the optimal unravelling is homodyne detection with θ the local oscillator phase.

CONCLUSIONS

- Feedback control problems for linear quantum systems can be treated using classical control theory.
- However, the constraints of quantum theory affect the basic structure of such problems.
- We have formulated a natural question — the optimal unravelling for a particular class of control problem — with no classical analogue.
- Moreover, these constraints also yield (under some assumptions) an efficient algorithm to answer this question.
- No doubt further fundamental aspects of quantum feedback control for linear systems await discovery.