INTRODUCTION TO REAL ANALYTIC GEOMETRY

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1. Analytic functions in several variables

1.1. **Summable families.** Let $(E, \| \|)$ be a normed space over the field $\mathbb{R}$ or $\mathbb{C}$, $\dim E < \infty$. Let $\{x_\alpha\}_{\alpha \in A}$ be a family (possibly infinite and even uncountable) of vectors in E. We say that this family is *summable* if there is $x \in E$ such that

$$\forall_{\epsilon>0}\, \exists_{F_\epsilon\, finite} \forall_{F_\epsilon \subset F\, finite}\, \|x - \sum_{\alpha \in F} x_\alpha\| < \epsilon.$$

We write in this case $x := \sum_{\alpha \in A} x_\alpha$, clearly x is unique.

We shall say that a collection $f_\alpha : Z \to E$, $\alpha \in A$ is *uniformly summable* if the family $\{f_\alpha(z)\}_{\alpha \in A}$ is summable for each $z \in Z$, moreover F_ϵ can be chosen independently of z.

Exercise 1.1. —The following conditions are equivalent:

(1) $\sum_{\alpha \in A} x_\alpha$ is summable
(2) $\sum_{\alpha \in A} \|x_\alpha\|$ is summable
(3) $\sup_{A \supset F - finite} \{\sum_{\alpha \in F} \|x_\alpha\|\} < +\infty$

Exercise 1.2. —Assume that $A = \bigcup_{\beta \in B} C_\beta$ is a disjoint union. Then $\sum_{\alpha \in A} x_\alpha$ is summable if and only if $c_\beta := \sum_{\alpha \in C_\beta} x_\alpha$ is summable for each $\beta \in B$ and $\sum_{\beta \in B} c_\beta$ is summable.

1.2. **Power series.** Let $\mathbb{K} = \mathbb{R}\, or\, \mathbb{C}$, for $z = (z_1, \dots, z_n) \in \mathbb{K}^n$ we denote $\|z\| = (|z_1|^2 \cdots + |z|^2)^{1/2}$.

Let $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, we recall standard notations : $\nu! := \nu_1! \cdots \nu_n!$, $\binom{\nu}{\mu} = \frac{\nu!}{\mu!(\nu-\mu)!}$ for $\mu \leq \nu$ in the partial order ($\mu \leq \nu \Rightarrow \mu_i \leq \nu_i$, $i = 1, \dots, n$).

$z^\nu := z_1^{\nu_1} \dots\ z_n^{\nu_n}$. For $a \in \mathbb{K}$ and $r = (r_1, \dots, r_n)$, $r_i > 0$ we denote by

$P(a, r) := \{z \in \mathbb{K}^n : |z_i - a_i| < r_i,\ i = 1, \dots, n\}$ the *poly-cylinder centered at a of poly-radius r.*

Exercise 1.3. Let $\theta = (\theta_1, \dots, \theta_n)$, $|\theta_i| < 1$, show that

$$\sum_{\nu \in \mathbb{N}^n} \theta^\nu = \frac{1}{(1-\theta_1) \cdots (1-\theta_n)}.$$

A family $a_\nu \in \mathbb{C}$, $\nu \in \mathbb{N}^n$ of complex numbers determines a formal power series $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$.

Lemma 1.4. *(Abel's Lemma) Let $a_\nu \in \mathbb{C}$, $\nu \in \mathbb{N}^n$, be a family of complex numbers (in other words a power series $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$ is given). Assume that there exists $b = (b_1, \dots, b_n) \in \mathbb{C}^*$ and $M > 0$ such that $|a_\nu b^\nu| \leq M$, for all $\nu \in \mathbb{N}^n$. Then $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$ is summable (we will say that the series converges) for any $z \in P(0, |b|)$, where $|b| = (|b_1|, \dots, |b_n|) \in \mathbb{C}^*$.*

Proof. Use Exercise 1.3. Note that actually the series converges absolutely.

□

Suppose that we are given a power series $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$. Put

$$P_l(z) := \sum_{\{\nu:|\nu|=l\}} a_\nu z^\nu,$$

this is a homogenous polynomial of degree l. Then (for a fixed $z \in \mathbb{C}^n$) the following conditions are equivalent:

(1) $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$ is summable,

(2) the series

$$\sum_{l=0}^{\infty} \left(\sum_{\{\nu:|\nu|=l\}} |a_\nu z^\nu| \right)$$

converges.

Note that the condition (2) above implies the series $\sum_{l=0}^{\infty} P_l(z)$ converges absolutely. By the Cauchy rule we obtain that

$$\gamma(z) := \limsup_{l\to\infty} \left(\sum_{\{\nu:|\nu|=l\}} |a_\nu z| \right)^{\frac{1}{l}} \leq 1,$$

which implies that $\sum_{\{\nu:|\nu|=l\}} P_l(z)$ converges absolutely. On the other hand if $\gamma(z) < 1$, then again by Cauchy's rule and the above equivalence we obtain that $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$ is summable. Thus we have obtained the following

Corollary 1.5. *If a series $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$ is summable for any $z \in P(0, r)$, then $\gamma(z) < 1$ for any $z \in P(0, r)$.*

This corollary enables as to associate to any power series the "sup" of poly-radiuses r on which we have $\gamma < 1$. We shall call such a $r \in \mathbb{R}^n_+$ *the radius of convergence*.

Suppose that we are given a (formal) power series $f = \sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$, let $k = 1, \dots, n$. Put

$$\frac{\partial f}{\partial z_k} := \sum_{\nu\in\mathbb{N}^n} \nu_k a_\nu z_1^{\nu_1} \cdots z_k^{\nu_k - 1} \cdots z_n^{\nu_n}.$$

Exercise 1.6. If a series f is summable in $P(0,r)$, then $\frac{\partial f}{\partial z_k}$ is also summable in $P(0,r)$. *Hint* : use the fact $\lim_{l\to\infty} l^{\frac{1}{l-1}} = 1$.

Definition 1.7. Let U be an open subset of $\mathbb{K}^n$, and let $f : U \to \mathbb{K}$ be a function. We say that f is *analytic at* $c \in U$ if there exist a power series $\sum_{\nu\in\mathbb{N}^n} a_\nu(z-c)^\nu$ *(called Taylor expansion of* f *at* c*)* and $r \in \mathbb{R}^n_+$ such that the series is summable in $P(c,r)$ and

$$f(z) = \sum_{\nu\in\mathbb{N}^n} a_\nu(z-c)^\nu,\ z \in P(c,r).$$

We say that f is *analytic in* U if f is analytic at any point of U. In the case $\mathbb{K} = \mathbb{C}$ analytic functions are rather called *holomorphic* .

Proposition 1.8. *Any analytic function* f *is infinitely many times* $\mathbb{K}$*-differentiable, moreover* $\frac{\partial f}{\partial z_k}$ *is again analytic.*

Proof. The result is classical for $n = 1$, so it is enough to use Exercise 1.6. We obtain also

$$\nu! a_\nu = \frac{\partial^\nu f}{\partial z^\nu}(c).$$

□

Theorem 1.9. *(Principle of analytic continuation) Let* U *be an open connected subset of* $\mathbb{K}^n$ *and* $f : U \to \mathbb{K}$ *an analytic function. Assume that at some* $c \in U$ *we have* $\frac{\partial^\nu f}{\partial z^\nu}(c) = 0$*, for all* $\nu \in \mathbb{N}^n$*. Then* $f \equiv 0$ *in* U*. In particular if* $f \equiv 0$ *in an open nonempty* $V \subset U$*, then* $f \equiv 0$ *in* U*.*

Proof. One can join any two points in U by a an arc piecewise parallel to coordinate axes. So we can apply the classical result in the case $n = 1$. □

Remark 1.10. It follows that, if U connected and $f : U \to \mathbb{K}$ is an analytic function such $f \not\equiv const$, then $\mathrm{Int} f^{-1}(0) = \emptyset$.

1.3. **Separate analyticty.**

Theorem 1.11. *(Osgood's lemma) Let* U *be an open subset of* $\mathbb{C}^n$ *and* $f : U \to \mathbb{C}$ *a locally bounded function which is holomorphic with respect to each variable separately. Then* f *is holomorphic in* U*.*

Remark 1.12. In fact according to a theorem of Hartogs the assumption that f is locally bounded is superfluous. But the proof of the Hartogs theorem requires a more advanced tools.

Proof. We may assume that $U = P(c,r)$ is a poly-cylinder. We shall proceed by the induction on n, the case $n = 1$ is trivial. We need a following

Lemma 1.13. *Let Ω be an open subset of $\mathbb{C}$ and let $g : \Omega \times [a, b] \to \mathbb{C}$ be a function. Assume that $g(z, t)$ is bounded, holomorphic with respect to z and contiuous with respect to t. Then the function*

$$h(z) := \int_a^b g(z, t)dt$$

is holomorphic in Ω.

Proof of the lemma. Let us fix $c \in \Omega$ and $B := B(c, \rho) \subset \Omega$ a disk such that its boundary $\partial B \subset \Omega$. Then by the Cauchy formula we may write

$$g(z, t) = \frac{1}{2\pi i} \int_{\partial B} \frac{g(\xi, t)}{\xi - z} d\xi,\ z \in B.$$

Hence g is locally uniformly continuous with respect to z, since g is bounded. Thus g is continuous (i.e. with respect to (z, t)-variables). So, by Fubini's theorem, for $z \in B$ we can write

$$h(z) = \int_a^b \left(\frac{1}{2\pi i} \int_{\partial B} \frac{g(\xi, t)}{\xi - z} d\xi \right) dt = \frac{1}{2\pi i} \int_{\partial B} \frac{1}{\xi - z} \left(\int_a^b g(\xi, t)dt \right) d\xi$$

That is $h(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{h(\xi)}{\xi - z} d\xi$, $z \in B$, which proves that h is holomorphic. So Lemma 1.13 follows.

To finish the proof of Theorem 1.11 we expand our function f on $P(c, r) \subset \mathbb{C}^n$ in the series

$$f(z) = \sum_{l=0}^{\infty} A_l(z')(z_n - c_n)^l$$

which converges absolutely, where $z' = (z_1, \dots, z_{n-1})$. Again thanks to Cauchy's formula we have

$$A_l(z') = \frac{1}{2\pi i} \int_{|\xi - c_n| = \rho} \frac{f(z', \xi)}{(\xi - c_n)^{l+1}} d\xi,$$

for any $0 < \rho < r_n$. Now, by Lemma 1.13, each function $A_l(z')$ is holomorphic with respect to each variable separately and locally bounded. Hence by the induction hypothesis each function $A_l(z')$ is actually holomorphic. Expanding $A_l(z')$ into a power series we find an expansion of f into a power series in the poly-cylinder $P(c, r)$. Exercise: check the convergence.

□

Remark 1.14. Clearly Osgood's lemma is false in the real case. For instance consider $f(x, y) = \frac{x^3}{x^2 + y^2}$, $f(0, 0) = 0$. Check that this function is continuous, analytic with respect to x and y, but not differentiable at the origin.

1.4. **Cauchy-Riemann equations and consequences.** Recall the classical basic functios about holomorphic functions of 1 variable. Let $f : U \to \mathbb{C}$ be function, where U is an open subset of $\mathbb{C}$, then the following conditions are equivalent :

(1) f is holomorphic in U;
(2) f is $\mathbb{C}$-differentiable at any point of U;
(3) f is $\mathbb{R}$-differentiable at any point of $a \in U$ and the (real) differential $d_a f : \mathbb{C} \to \mathbb{C}$ is actually $\mathbb{C}$-linear;
(4) f is $\mathbb{R}$-differentiable and the Cauchy -Riemann equations are satisfied:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$
where $u = Re\, f$ and $v = Im\, f$.

Let us recall that if Y and Z are vector spaces over $\mathbb{C}$ then they carry a unique structure of vector spaces over $\mathbb{R}$. Consider an $\mathbb{R}$-linear map $\varphi : Z \to Y$, then φ is $\mathbb{C}$- linear if and only if
$$\varphi(iz) = i\varphi(z),\ z \in Z.$$

Now we can state the main theorem about holomorphic functions in several variables.

Theorem 1.15. *Let $f : U \to \mathbb{C}$ be function, where U is an open subset of $\mathbb{C}^n$. Then f is holomorphic in U if and only if f is continuous (even merely locally bounded) and $\frac{\partial f}{\partial z_k}$, $k = 1, \ldots, n$ (complexes derivatives) exists at any point in U.*

Proof. Apply Osgood's lemma and use the above results in 1 variable. □

Definition 1.16. Let U be a an open subset of $\mathbb{C}^n$, we say that a map $F = (f_1, \ldots, f_k) : U \to \mathbb{C}^k$ *holomorphic* if each f_j is holomorphic.

Proposition 1.17. *A map $F = (f_1, \ldots, f_k) : U \to \mathbb{C}^k$ is holomorphic if and only if F is C^1 in the real sense and the differential $d_a f$ is $\mathbb{C}$-linear at every $a \in \mathbb{C}$.*

Proof. This is an immediate consequence of Theorem 1.15. □

Thus we obtain the following basic properties of holomorphic maps.

Corollary 1.18.

(1) *If f and g are holomorphic then also , $(f + g)$,(fg), $(\frac{f}{g})$ (where defined) are holomorphic.*
(2) *If G and F are holomorphic maps, then $G \circ F$ is holomorphic.*
(3) *If F is holomorphic such that F^{-1} exists and $d_a F$ is an isomorphism for each $a \in U$ (the last assumption is actually superfluous), then F^{-1} is holomorphic.*
(4) *Implicit function theorem holds in the holomorphic setting.*

Note that the explicit and direct estimates in the above statements for poly-radius of convergence are not obvious at all.

1.5. **Real analytic functions.** Let W be an open subset in $\mathbb{R}^n$ and $f : W \to \mathbb{R}$ an analytic function. This means that for any $a \in W$ the function f can be expanded in a power series in $P_{\mathbb{R}}(a,r)$ for some $r = r(a) \in \mathbb{R}^n_+$, where

$$P_{\mathbb{R}}(a,r) := \{z \in \mathbb{R}^n : |z_i - a_i| < r_i,\ i = 1, \dots, n\} = \mathbb{R}^n \cap P_{\mathbb{C}}(a,r).$$

Here $P_{\mathbb{C}}(a,r) := \{z \in \mathbb{C}^n : |z_i - a_i| < r_i,\ i = 1, \dots, n\}$.

Proposition 1.19. *There exist an open set* $\widetilde{W} \subset \mathbb{C}^n$*,* $W \subset \widetilde{W}$ *and holomorphic function* $\widetilde{f} : \widetilde{W} \to \mathbb{C}$ *such that* $\widetilde{f}|_W = f$*. Moreover* $(\widetilde{W}, \widetilde{f})$ *are unique in the following sense. If* $\widetilde{W}_1 \subset \mathbb{C}^n$ *is an open set and* $\widetilde{f}_1 : \widetilde{W}_1 \to \mathbb{C}$ *a holomorphic function such that* $\widetilde{f}_1|_W = f$*, then there exists an open set* $U \subset \mathbb{C}^n$*,* $W \subset U$ *and such that* $\widetilde{f}_1|_U = \widetilde{f}|_U$*.*

We shall call the holomorphic function $\widetilde{f} : \widetilde{W} \to \mathbb{C}$ *complexification* of f.

Proof. Put $\widetilde{W} = \bigcup_{a \in W} P_{\mathbb{C}}(a, r(a))$. the function $\widetilde{f} := \bigcup_{a \in W} \widetilde{f}_a$. Here $\widetilde{f}_a$ is the holomorphic functionin $P_{\mathbb{C}}(a, r(a))$ defined by the power series obtained at a. We leave as exercise details to be checked: that $\widetilde{f}$ is well defined and the second part of the statement. Hint: use analytic continuation theorem. $\square$

Corollary 1.20. *In Corollary 1.18 we may replace "holomorphic" by "real analytic".*

Let U be an open subset of $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ a holomorphic function. We put $U^c = \{z \in \mathbb{C}^n : \overline{z} \in U\}$, where $\overline{z} := (\overline{z}_1, \dots, \overline{z}_n)$ and $\overline{f}(z) := \overline{f(\overline{z})}$. Note that $\overline{f}$ is actually holomorphic (check Cauchy-Riemann equations). Observe however that the function $z \mapsto \overline{f(z)}$ is not holomorphic if $f \neq const$.

Proposition 1.21. *Let* U *be an open subset of* $\mathbb{C}^n$ *and* $f : U \to \mathbb{C}$ *a holomorphic function. Then* $f(x) \in \mathbb{R}^n$*, for all* $x \in U \cap \mathbb{R}^n$ *if and only if* $f = \overline{f}$ *in a neighborhood of* $U \cap \mathbb{R}^n$*.*

Proof. Prove that both conditions are equivalent to the fact that all coefficients of the Taylor expansion f at a point in $U \cap \mathbb{R}^n$ are real. $\square$

1.6. **Riemann extension theorem.** Let U be an open subset of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ and let Z be a closed subset of U such that for any $z' \in C^{n-1}$ the set $(\{z'\} \times \mathbb{C}) \cap Z$ consists only of isolated points. We will say that Z is *negligible* .

Exercise 1.22. Assume that U is connected and that Z is negligible in U Show that $U \setminus Z$ is also connected.

Theorem 1.23. *(Riemann extension theorem) Assume that $f : U \setminus Z \to \mathbb{C}$ is a holomorphic bounded function and that Z is negligible . Then f extends to a unique holomorphic function on U.*

Proof. Let $z = (z', z_0) \in Z$, since Z is closed there exist $\delta, \varepsilon > 0$ such that

$$Z \cap (B(z', \delta) \times \{\xi : |\xi - z_0| = \varepsilon\}) = \emptyset,$$

where $B(z', \delta)$ is an open disk in $\mathbb{C}^{n-1}$. Let us define

$$\widetilde{f}(w, t) := \frac{1}{2\pi i} \int_{|\xi - z_0| = \varepsilon} \frac{f(w, \xi)}{\xi - t} d\xi,$$

for $w \in B(z', \delta)$, $t \in B(z_0, \varepsilon)$. Note that, by Lemma 1.13 $\widetilde{f}$ is holomorphic with respect to each w_j-variable, it is holomorphic with respect to t-variable by the classical result, moreover it is bounded. Hence by Osgood's lemma (Theorem 1.11) our function $\widetilde{f}$ is holomorphic in $B(z', \delta), \times B(z_0, \varepsilon)$. Check that $\widetilde{f} = f$ outside Z and prove the uniqueness. □

2. Weierstrass Preparation Theorem

2.1. **Symmetric polynomials and Newton sums.** Let A be a commutative ring with unit. We say that a polynomial $P \in A[X_1, \ldots, X_k]$ is *symmetric* if for any permutation τ we have

$$P(X_{\tau(1)}, \ldots, X_{\tau(k)}) = P(X_1, \ldots, X_k).$$

Let us write

$$(T - X_1) \cdots (T - X_k) = T^k + \sigma_1 T^{k-1} + \cdots + \sigma_k,$$

where

$$\sigma_j := (-1)^j \sum_{\nu_1 < \cdots < \nu_j} X_{\nu_1} \cdots X_{\nu_j}$$

Recall that σ_j is called *j-th elementary symmetric polynomial.* If $\xi_1, \ldots, \xi_k$ are all the roots of $P = Z^k + a_1 Z^{k-1} + \cdots + a_k$, then we have *Viéte formulas*

$$a_j = \sigma_j(\xi_1, \ldots, \xi_n)$$

Important symmetric polynomials are *Newton sums*

$$s_l := \sum_{i=1}^{k} X_1^l + \cdots + X_k^l$$

Lemma 2.1. *There are polynomials $R_j \in \mathbb{Z}[Y_1, \ldots, Y_n]$ such that*

$$\sigma_j = R_j(s_1, \ldots, s_k)$$

A celebrate theorem on symmetric polynomials claims the following.

Theorem 2.2. *Let A be a commutative ring with unit and let $P \in A[X_1, \dots, X_k]$ be symmetric polynomial, then there exist a unique $Q \in A[Y_1, \dots, Y_k]$ such that*

$$P = Q(\sigma_1, \dots, \sigma_n)$$

If the ring A contains $\mathbb{Q}$, then there exist a unique $R \in A[Y_1, \dots, Y_k]$ such that

$$P = R(s_1, \dots, s_k).$$

2.2. **Generalized discriminants.** Let us consider a generic polynomial

$$P_c(z) = z^k + c_1 z^{k-1} + \dots + c_k$$

where $z \in \mathbb{C}$ and $c = (c_1, \dots, c_k) \in \mathbb{C}^k$. Put

$$W_s := \{c \in \mathbb{C}^k : P_c(z) \text{ has at most s distinct complex roots } \}.$$

Let $K = \{1, \dots, k\}$ and put

$$\mathcal{D}_s(z_1, \dots, z_k) = \sum_{J \subset K \# J = k-s} \quad \prod_{\mu<\nu;\, \mu,\nu \in J} (z_\mu - z_\nu)^2 \qquad , \quad s = 0, \dots, k-1$$

Since $\mathcal{D}_s(z_1, \dots z_k)$ is a symmetric polynomial, by Theorem 2.2 there exists $D_s \in \mathbb{C}[c_1, \dots, c_k]$ such that $\mathcal{D}_s = D_s \circ \sigma$ where $\sigma = (\sigma_1, \dots, \sigma_k)$. We call D_s, $s = 0, \dots, k-1$ *generalized discriminants of* P.

Lemma 2.3.

$$W_s = \{c \in \mathbb{C}^k : D_0(c) = \dots = D_{k-s-1}(c) = 0\}$$

Proof. Indeed, if $c \in W_s$ and $\xi = (z_1, \dots, z_k)$ are all the roots (with possible repetition) of $P_c(z)$, then $\#\{z_1, \dots, z_k\} \le s$, hence

$$\mathcal{D}_0(\xi) = \dots = \mathcal{D}_{k-s-1}(\xi) = 0,$$

which implies $D_0(c) = \dots = D_{k-s-1}(c) = 0$.

Let $c \in \mathbb{C}^k$ be such that $D_0(c) = \dots = D_{k-s-1}(c) = 0$. Let $\xi = (z_1, \dots, z_k)$ the complete sequence of roots of P_c. Assume that $c \notin W_s$, $s+1 \le \#\{z_1, \dots, z_k\} = l$. Let $z_1, \dots z_t$ be all distinct l roots t of $P_c(z)$. Then

$$\mathcal{D}_j(z_1, \dots, z_k) = D_j(c) = 0 \qquad \text{if} j = 0, 1, \dots k-s-1.$$

Since $k - l \le k - s - 1$,

$$0 = \mathcal{D}_{k-t}(z_1, \dots, z_k) = \prod_{\mu<\nu;\, \mu,\nu \in \{1,\dots,t\}} (z_\mu - z_\nu)^2 \,,$$

which is absurd.

□

Note that $D := D_{k-2}$ is the *discriminant* of P, we have

$$D = \prod_{\mu<\nu} (z_\mu - z_\nu)^2 = \pm \prod_{\nu=1} P'(z_\nu).$$

In particular $D(c) \neq 0$ if and only if all roots of P_c are simple.

Corollary 2.4. *Each W_s is algebraic.*

2.3. **Continuity of roots.** Let us consider a generic polynomial

$$P_c(z) = z^k + c_1 z^{k-1} + \cdots + c_k,$$

where $z \in \mathbb{C}$ et $c = (c_1, \ldots, c_k) \in\ C^k$. Suppose that for some $r > 0$ we have $|c_j| \leq r^j$, $j = 1, \ldots, k$, then

$$P(z) = 0 \ \Rightarrow |z| \leq 2r.$$

Indeed we have

$$|z^k + c_1 z^{k-1} + \cdots + c_k| \geq |z^k| \left(1 - \frac{r}{|z|} \cdots - \frac{r^k}{|z^k|}\right) > 0,$$

if $\frac{r}{|z|} \leq \frac{1}{2}$.

The following notion from general topology will be important in the next paragraphs.

Definition 2.5. A continuous map $f : X \to Y$, between two topological spaces is said to be proper, if for any compact $K \subset Y$ the inverse image $f^{-1}(K)$ is compact.

Proposition 2.6. *If X and Y are locally compact (i.e. every point has a compact neighborhood) then f is proper if and only for each $y \in Y$ there exists a neighborhood V such that $f^{-1}(V)$ is relatively compact (i.e. its closure is compact).*

Proof. Exercise □

Recall that we have a natural Viéte map

$$\sigma = (\sigma_1, \ldots, \sigma_n) : \mathbb{K}^n \to \mathbb{K}^n.$$

So we have proved

Proposition 2.7. *The map $\sigma : \mathbb{K}^n \to \mathbb{K}^n$ is proper and surjective if $\mathbb{K} = \mathbb{C}$.*

Theorem 2.8. *Let*

$$P_c(z) = z^k + c_1 z^{k-1} + \cdots + c_k,$$

where $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$. Let $z_1, \ldots, z_s$ be all distinct roots of P_c. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that:

if $c' \in \mathbb{C}^k$, $|c' - c| < \delta$, and $z' \in \mathbb{C}$ such that $P_{c'}(z') = 0$, then $|z' - z_j| < \varepsilon$ for some $j = 1, \ldots, s$.

Proof. Let $r > 0$ be such that $|c_j| \leq r^j$, $j = 1, \ldots, k$, and put $R := 2r$. The set

$$K := \overline{B}(0, R) \setminus \bigcup_{j=1}^{s} B(z_j, \varepsilon)$$

is compact and nonempty if r is large enough. The map $(w, c) \mapsto |P_c(w)|$ is continuous and strictly positive on the compact $z \times K$, hence it is also strictly positive on $\overline{B}(z,\delta)\times K$ if $\delta>0$ is small enough. Decreasing, if necessary, δ we may assume that $P_{c'}$ has no roots outside $\overline{B}(0,R)$, so the theorem follows. □

2.4. **Weierstrass preparation theorem.** Let U an open neighborhood of $0\in\mathbb{C}^n$, we write $z=(z',z_n)\in\mathbb{C}^{n-1}\times\mathbb{C}=\mathbb{C}^n$. Let $f:U\to\mathbb{C}$ be a holomorphic function. We shall say that f is *k-regular at* 0, if

$$\frac{\partial^j f}{\partial z_n^j}(0)=0,\ j=1,\dots,k-1\ and\ \frac{\partial^k f}{\partial z_n^k}(0)\neq 0.$$

In other words f is k-regular if $z_n\mapsto f(0,z_n)=z_n^k\varphi(z_n)$ with φ holomorphic and $\varphi(0)\neq 0$. We denote

$$P(\varepsilon,\delta):=P'(0,\varepsilon)\times B(0,\delta),$$

where $P'(0,\varepsilon)$ is a poly-disk of radius ε. Let $c_j:P'(0,\varepsilon)\to\mathbb{C}$, $j=1,\dots,k$ be holomorphic functions, $c_j(0)=0$. We call

$$P(z',z_n)=z_n^k+\sum_{j=1}^{k}c_j(z')z_n^{k-j}$$

a Weierstrass polynomial.

Remark 2.9. If $f=\sum_{l\geq l_0}P_l$ is the expansion into series of homogenous polynomials, $P_{l_0}\neq 0$, then any line L such that $L\not\subset P_{l_0}^{-1}(0)$ can be chosen as z_n -axis and f will be l_0-regular.

Theorem 2.10. *Let U an open neighborhood of $0\in\mathbb{C}^n$.Let $f:U\to\mathbb{C}$ be a holomorphic function which is k-regular at 0. Then there exists $\varepsilon,\delta>0$ a Weierstrass polynomial P in the poly-disk $P(\varepsilon,\delta)$ and holomorphic function φ nowhere vanishing in $P(\varepsilon,\delta)$ such that*

$$f(z',z_n)=\varphi(z',z_n)P(z',z_n)$$

for $(z',z_n)\in P(\varepsilon,\delta)$. Moreover

(1) *P and φ are unique, P will be called the Weierstrass polynomial associated to f,*
(2) *if f is real then P and φ are also real.*

Proof. **The uniqueness.** Suppose that $f=\varphi P=\varphi_1P_1$ in some polydisk $P(\varepsilon,\delta)$. By the continuity of roots may decrease δ in such way that if $z'\in B'(0,\varepsilon)$, $z_n\in\mathbb{C}$ and $P(z',z_n)=0$, then $|z_n|<\varepsilon$. We may also assume this property for the polynomial P_1. So for $z'\in B'(0,\varepsilon)$ two univariate monic polynomials

$$z_n\mapsto P(z',z_n),\ z_n\mapsto P_1(z',z_n)$$

have the same roots and with same multiplicities. Hence they are equal. It follows that $\varphi=\varphi_1$ in $P(\varepsilon,\delta)\setminus P^{-1}(0)$ which is dense in $P(\varepsilon,\delta)$, so $\varphi=\varphi_1$ in $P(\varepsilon,\delta)$.

Reality. The function f is real if and only if $f = \overline{f}$, hence

$$\varphi P = f = \overline{\varphi}\overline{P}$$

By the uniqueness we obtain $P = \overline{P}$, $\varphi = \overline{\varphi}$. So P and φ are real.

Existence. Le us fix ε such that $z_n \mapsto f(0, z_n)$ has no zeros in the punctured disk $\{0 < |z_n| \leq \varepsilon\}$. Recall that k is the multiplicity of this function at $0 \in \mathbb{C}$. By the continuity argument there exists $\delta > 0$ such that if $z' \in P'(0, \delta)$, then $z_n \mapsto f(z', z_n)$ has no zeros in the circle $\{|z_n| = \varepsilon\}$.

According to the theorem of Rouché $z_n \mapsto f(z', z_n)$ has k zeros in the disk $\{|z_n| < \varepsilon\}$. Let us denote those zeros by $w_1(z'), \ldots, w_k(z')$. Put

$$P(z', z_n) := (z_n - w_1(z') \cdots (z_n - w_k(z')) = z_n^k + c_1(z')z_n^{k-1} + \cdots + c_k(z'),$$

with $c_j(z') = \sigma_j(w_1(z'), \ldots, w_k(z'))$. To show that P is a Weierstrass polynomial it is enough to check that each $c_j(z')$ is holomorphic. By Theorem 2.2, it is enough to show that

$$S_j = s_j(w_1(z'), \ldots, w_k(z')) = w_1(z')^j + \cdots + w_k(z'))^j$$

are holomorphic for $j = 1, \ldots, k$. According to the theorem on logarithmic residus we have

$$S_j(z') := \frac{1}{2\pi i} \int_{|z_n|=\varepsilon} z_n^j \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} dz_n,$$

By Theorem 1.11 and Lemma 1.13 functions $S_j(z')$ are holomorphic. To conclude note that

$$\varphi(z', z_n) = \frac{f(z', z_n)}{P(z', z_n)}$$

is holomorphic and bounded in the complement of zeros of P. So by Riemann's Extension Theorem φ is actually holomorphic in $P(\varepsilon, \delta)$. Finally note that φ has no zeros in $P(\varepsilon, \delta)$ since zeros of f and P have the same multiplicities (with respect to z_n).

□

Remark 2.11. If $z_n \mapsto f(0, z_n) \not\equiv 0$, then f is k regular for some k. Hence, for z' close enough $0 \in \mathbb{C}^{n-1}$ the function $z_n \mapsto f(z', z_n)$ has at most k zeros in $B(0, \varepsilon)$. Assume now the contrary that $z_n \mapsto f(0, z_n) \equiv 0$ but $f \not\equiv 0$.

Can we bound the number of zeros (close to the origin) of $z_n \mapsto f(z', z_n)$? (provided that $z_n \mapsto f(z', z_n) \not\equiv 0$)

The answer is positive, the first (and forgotten for some time) solution (algebraic) was given by Bautin (1939), the second (geometric) is due to Gabrielov (1968) and become a milestone in the real analytic (more precisely subanalytic) geometry.

2.5. **Weierstrass division theorem.**

Theorem 2.12. *Let U an open neighborhood of $0 \in \mathbb{C}^n$. Let $f; g : U \to \mathbb{C}$ be two holomorphic functions. Assume that f is k-regular at 0. Then there exists $\varepsilon, \delta > 0$ such that in the poly-disk $P(\varepsilon, \delta)$ we have*

$$g = Qf + R$$

for $(z', z_n) \in P(\varepsilon, \delta)$, with R holomorphic in $P(\varepsilon, \delta)$ of the form

$$R(z', z_n) = \sum_{j=1}^{d} a_j(z') z_n^{k-j},$$

where $d < k$ and $a_j : P'(0, \varepsilon) \to \mathbb{C}$ are holomorphic. Moreover

1. *Q and R are unique (that is their Taylor series at 0 are unique),*
2. *if f and g are real then Q and R are also real.*

Proof. **Uniqueness.**

Assume that $Qf + R = g = Q_1 f + R_1$, then $0 = (Q - Q_1)f + (R - R_1)$. Hence it is sufficient to show that if $g \equiv 0$ then $Q \equiv 0$ and $R \equiv 0$. Indeed, for $z' \in \mathbb{C}^{n-1}$ close enough to 0 the function $z_n \mapsto f(z'; z_n)$ has k zeros in $\{|z_n| < \varepsilon\}$, this follows from Weierstrass Preparation Theorem. Hence $z_n \mapsto R(z', z_n)$ must have at least k roots. But degree of R is less than k so $R \equiv 0$, which implies $Q \equiv 0$.

Reality . The same argument as in the proof of Theorem 2.10.

Existence. By Preparation Theorem 2.10 we may assume that f is a Weierstrass polynomial. Also we may assume that f and g are holomorphic in a neighborhood of $\overline{P}(\varepsilon, \delta)$, moreover that $z' \in P'(0, \varepsilon)$, $f(', z_n) = 0 \Rightarrow |z_n| < \varepsilon$.

Hence the function

$$Q(z', z_n) := \frac{1}{2\pi i} \int_{\{|\xi| = \varepsilon\}} \frac{g(z', \xi)}{f(z', z_n)} \frac{1}{\xi - z_n} d\xi,$$

is holomorphic in $P(\varepsilon, \delta)$, by By Theorem 1.11 and Lemma 1.13. On the other hand

$$g(z', z_n) = \frac{1}{2\pi i} \int_{\{|\xi| = \varepsilon\}} \frac{g(z', \xi)}{\xi - z_n} d\xi,$$

in $P(\varepsilon, \delta)$, so

$$(g - Qf)(z', z_n) = \frac{1}{2\pi i} \int_{\{|\xi| = \varepsilon\}} \frac{g(z', \xi)}{f(z', z_n)} \Gamma(z', \xi, z_n) d\xi,$$

where

$$\Gamma(z', \xi, z_n) := \frac{f(z', \xi) - f(z', z_n)}{\xi - z_n}.$$

Note that $z_n \mapsto \Gamma(z', \xi, z_n)$ is a polynomial of degree less than k, the coefficients are actually holomorphic in $P(\varepsilon, \delta)$. Thus $R := Qf - g$ is a polynomial in z_n of degree less than k, with coefficients holomorphic in $P'(0,\delta)$. □

Remark 2.13. The division theorem holds also for formal power series, also in some refined version.

2.6. **Decomposition of a Weierstrass polynomial into irreducible factors.** We change a bit the notation. Let U an open subset connected subset of $\mathbb{C}^n$, we denote $\mathcal{O}(U)$ the ring of holomorphic functions on U. We consider a monic polynomial

$$P(u,z) = z^k + \sum_{j=1}^{k} c_j(u) z_n^{k-j}$$

with $c_j \in \mathcal{O}(U)$. Our goal is to show

Theorem 2.14. *There are unique monic irreducible polynomials $Q_1, \cdots, Q_l \in \mathcal{O}(U)[z]$ and integers $\nu_1, \ldots, \nu_l$ such that*

$$P = Q_1^{\nu_1} \cdots Q_l^{\nu_l}.$$

Proof. We shall use generalized discriminants D_s. For $s = 0, \ldots, k-1$ we put

$$\Delta_s(u) = D_s(c_1(u), \ldots, c_k(u))$$

Hence Δ_s are holomorphic in U. Since U is connected we have two possibilities: either $\Delta_s \equiv 0$ or $\operatorname{Int} \Delta_s^{-1}(0) = \emptyset$. Let $r \leq k$ be such a integer that

$$\Delta_0 \equiv \cdots \Delta_{k-r-1} \, and \, \Delta_{k-r} \not\equiv 0.$$

Let $\Omega := U \setminus \Delta_{k-r}^{-1}(0)$. According to Lemma 2.3 for any $a \in \Omega$ polynomial $z \mapsto P(a,z)$ has exactly r complex roots which we denote by $\xi_1(a), \ldots, \xi_r(a)$. Note that there is no natural way to label these roots, they should be seen as a set. However if we fix arbitrary an order as above, then we have, by the continuity of roots and Rouché's theorem the following :

Lemma 2.15. *One can choose continuously roots ξ_j in a neighborhood of any point $a \in \Omega$.*

As consequence each root ξ_j has a fixed multiplicity ν_j. It means that for b close enough to a

$$\frac{\partial^{\nu_j - 1} P}{\partial z^{\nu_j - 1}}(b, \xi_j(b)) = 0 \, and \, \frac{\partial^{r_j} P}{\partial z^{r_j}}(b, \xi_j(b)) \neq 0.$$

Hence applying Implicit Function Theorem we obtain

Lemma 2.16. *One can choose holomorphically roots ξ_j in a neighborhood of any point $a \in \widetilde{U}$.*

Let $Z := P^{-1}(0) \cap (\Omega \times \mathbb{C})$ and let $\pi : Z \to \Omega$ denote the projection. It follows from Lemma 2.15 that π is a finite covering. Let $Z_1, \ldots, Z_l$ be connected components, then (the restriction) $\pi : Z_i \to \Omega$ is again a finite (k_i-sheeted) covering (see Exercise 3.4 for the definition). Let us assume that $\xi_1(a), \ldots, \xi_{k_i}(a)$ are the roots $z \mapsto P(a, z)$ which correspond to the component Z_i. We put, for any $a \in \Omega$

$$c_q(a) = \sigma_q(xi_1(a), \ldots, \xi_{k_i}(a)),\ q = 1, \ldots, k_i.$$

Note that each c_q is holomorphic and locally bounded function, hence by the Riemann Extension Theorem it extends to a holomorphic function on U. So we can now define irreducible factors.

$$Q_i(u, z) := z^{k_i} + \sum_{q=1}^{k_i} c_q(u) z^{k_i - q}$$

We leave the uniqueness of the decomposition as an exercise.

□

Exercise 2.17. Show that P is irreducible if an only if its discriminant is non-identically vanishing in U.

2.7. **The theorem of Puiseux.**

Theorem 2.18. *Let*

$$P(u, z) = z^k + \sum_{j=1}^{k} c_j(u) z_n^{k-j},$$

where c_j are holomorphic functions in the disk $B(0, \delta) \subset \mathbb{C}$. Assume that P is irreducible and that the discriminant of P vanishes only at $0 \in \mathbb{C}$. Then there exists a holomorphic function $h : B(0, \delta^{1/k})\mathbb{C}$ such that

$$P(u^k, z) = \prod_{j=0}^{k-1} (z - h(\theta_j u)),$$

where $\theta_0, \ldots, \theta_{k-1}$ are the roots of unity of order k.

The idea of the proof: Consider $Z = P^{-1}(0) \setminus \{0\} \times \mathbb{C}$, the canonical projection $\pi : Z \to B^* := B(0, \delta) \setminus \{0\}$ is a k-sheeted covering. Since P is irreducible Z is connected. Let $B_k^* := B(0, \delta^{1/k}) \setminus \{0\}$. Now consider the map

$$\varphi : B_k^* \ni u \mapsto u^k \in B^*$$

this is also is a k-sheeted covering. Finally study the pull back of π by $\Phi(u, t) = (u^k, t)$, and show that $\Phi^{-1}(Z)$ has k connected components. Conclude the result.

3. More exercises

Exercise 3.1. Maximum Principle. Let $U \subset \mathbb{C}^n$ be open and connected. Let $f : U \to \mathbb{C}$ be a holomorphic function. Assume that there exists $a \in U$ such that

$$|f(a)| = \sup_{z \in U} |f(z)|,$$

then f is constant. More generally, show that if f is non-constant, then f is open. Is the last statement true for holomorphic maps $F : U \to \mathbb{C}^n$? (Consider $F(x, y) = (x, xy)$.)

Exercise 3.2. Let $P(u, z) = z^k + c_1(u)z^{k-1} + \cdots + c_k(u)$, with c_j holomorphic in an open and connected $U \subset \mathbb{C}^n$. Put $Z := P^{-1}(0)$ and let $\pi : U \times \mathbb{C} \to U$ stand for the canonical projection. Show that $\pi|_Z$ - the restriction of π to Z, is open and proper. Which of these properties remain true in the real case ?

More generally (in the complex case) we can consider $\Omega = U \times V$, where $V \subset \mathbb{C}$ is open. What can be said about $\pi|_{Z \cap \Omega}$?

Exercise 3.3. Let $U \subset \mathbb{C}^n$ be open and connected. Let $f : U \to \mathbb{C}$ be a holomorphic non-constant function. Show that $U \setminus f^{-1}(0)$ is connected.

Exercise 3.4. Let M and N be two locally connected topological spaces. Recall that a continuous map $\varphi : M \to N$ is *covering*, if for each $y \in N$ there exists a neighborhood V of y such that $\varphi^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha \neq \emptyset$ (a disjoint union of open sets) such that for each $\alpha \in A$ the map $\varphi|_{U_\alpha} : U_\alpha \to V$ is a homeomorphism. Assume that $\varphi : M \to N$ is a *finite covering* (i.e all fibers are finite). Prove the following:

(1) if N is connected then all fibers have the same cardinality k, we will say that *the covering is k-sheeted*;
(2) if N is connected and $\widetilde{M} \subset M$ is an open and closed subset of M (e.g. $\widetilde{M}$ may be a connected component of M) , then $\varphi|_{\widetilde{M}} : \widetilde{M} \to N$ is again a covering.
(3) let $\gamma : [0, 1] \to N$ be a continuous arc, let $x_0 \in M$ be such a point that $\varphi(x_0) = \gamma(0)$, then there exists a unique $\widetilde{\gamma} : [0, 1] \to M$ such that $\widetilde{\gamma}(0) = x_0$ and $\varphi \circ \widetilde{\gamma} = \gamma$.

Exercise 3.5. Let M and N be two locally compact topological spaces, show that $\varphi : M \to N$ is a finite covering if and only if φ is proper local homeomorphism.

Exercise 3.6. Chow's Theorem for hypersurfaces. Let $U \subset \mathbb{C}^n$ be open and convex neighborhood of $0 \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be a holomorphic function, denote $Z := f^{-1}(0)$. Assume that Z is *homogenous* that means: $z \in Z$, $|t| \leq 1 \Rightarrow tz \in Z$, equivalently that for any complex vector line $L \subset \mathbb{C}^n$ we have either $L \cap Z = L \cap U$ or $L \cap Z = \{0\}$. Show that Z is actually algebraic, precisely that there exists a homogenous polynomial $g =: \mathbb{C}^n \to \mathbb{C}$ such that $Z = U \cap g^{-1}$.