Optimal portfolio for CRRA utility functions where risky assets are exponential additive processes

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Outline

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The assets

Consider a portfolio with a locally riskless asset $B \equiv 1$, and risky assets S^i, $i = 1, \ldots, n$, which evolves as an exponential additive processes

$$dS_t^i = S_{t-}^i \; dR_t^i \tag{1}$$

where R^i, $i = 1, \ldots, n$, are the components of the additive process $R = (R^1, \ldots, R^n)$ with dynamics

$$dR_t = \mu(t)dt + \sigma(t)dw_t + \int_{\mathbb{R}^\ltimes} x(N(dx, dt) - \nu_t(dx)dt)$$

with $\mu = (\mu_1, \ldots, \mu_n) : [0, T] \to \mathbb{R}^n$, $\sigma = (\sigma_{ij})_{ij} : [0, T] \to \mathbb{R}^{n \times d}$ measurable functions, $w = (w^1, \ldots, w^d)$ a d-dimensional Brownian motion and $N(dx, dt)$ a Poisson random measure N on $\mathbb{R}^n$ with compensating measure $\nu_t(dx)dt$.

Compact notation

We can write the dynamics of $S_t = (S_t^1, \dots, S_t^n)$ in a more compact vectorial notation as

$$dS_t = \mathrm{diag}(S_{t-})\ dR_t$$

where $diag(v)$ denotes the diagonal matrix in $\mathbb{R}^{n \times n}$ having in the principal diagonal the elements of the n-dimensional vector v.

Positivity of prices

In order to guarantee that the price of the n assets stays a.s. positive for all $t \in [0, T]$, we assume that

$$\mathrm{supp}(\nu_t) \subseteq X := \{x \in \mathbb{R}^n | x_i \geq -1 \quad \forall i = 1, \dots, n\}$$

and that $\nu_t(\partial X) = 0$ for all $t \in [0, T]$.

Remark: When dealing with exponential additive processes, it is usual to write them as e^{L_t}, with L a suitable additive process. Equation (1) has solution

$$S_t^i = e^{R_t^i - \frac{1}{2}\int_0^t \sum_{j=1}^n \sigma_{ij}^2(s)\ ds} \prod_{0<s\leq t} (1 + \Delta R_s^i) e^{-\Delta R_s^i}$$

with $\Delta R_s^i := R_s^i - R_{s-}^i$. Under the assumptions above, $1 + \Delta R_t^i > 0$ $\mathbb{P}$-a.s., so the S^i, $i = 1, \dots, n$, are strictly positive processes and can be written as $e^{L_t^i}$, with L^i additive processes.

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Finite variance

We also want the assets to have finite variance: a sufficient condition for this to hold true is to impose that

$$\int_0^T \left(\|\mu(t)\|_n + \|\sigma(t)\|^2_{n\times d} + \int_{\mathbb{R}^n} \|x\|_n^2 \, \nu_t(dx) \right) \, dt < +\infty \quad (2)$$

where $\|x\|_n^2 := \sum_{i=1}^n x_i^2$ and $\|A\|^2_{n\times d} := \sum_{i=1}^n \sum_{j=1}^d A_{ij}^2$. In fact, in this case the process R has finite variance, and thus one can prove that $\mathbb{E}[\|S_t\|_n^2] < +\infty$ for all $t \in [0, T]$, i.e., the risky assets S^i, $i = 1, \ldots, n$ have all finite variance.

The investment portfolio

Let now $h_t := (h_t^1, \dots, h_t^n)$ be the proportions of the portfolio invested respectively in the assets $(S^1, \dots, S^n)$ at time t; then the dynamics of the portfolio value V^h can be written as

$$dV_t^h = \sum_{i=1}^{n} \frac{V_{t-}^h h_{t-}^i}{S_{t-}^i} dS_t^i = V_{t-}\langle h_{t-}, dR_t \rangle \quad (3)$$

where $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$ denotes the scalar product in $\mathbb{R}^n$.
In order for this dynamics to be well defined, V^h must stay $\mathbb{P}$-a.s. positive for all $t \in [0, T]$.

Positivity of portfolio

As before, V^h as solution of Equation (3) can be written as

$$V_t^h = e^{\langle h_{t-}, dR_t\rangle - \frac{1}{2}\int_0^t \langle h_{s-}, \Sigma(s) h_{s-}\rangle \; ds} \prod_{0<s\leq t} (1+\langle h_{s-}, \Delta R_s\rangle) e^{-\langle h_{s-}, \Delta R_s\rangle}$$

with $\Sigma(t) = (a_{ij}(t))_{ij} := \sigma(t)\sigma^T(t)$. To require V positive is thus equivalent to requiring that $\langle h_{s-}, \Delta R_s\rangle > -1$ $\mathbb{P}$-a.s. for all $t \in [0, T]$, i.e. that

$$h_t \in H_t := \{h \in \mathbb{R}^n \mid \langle h, x\rangle > -1 \quad \nu_t(dx)\text{-a.s.} \} \tag{4}$$

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Example 1: unbounded jumps

If the jumps of all the risky assets are unbounded in both directions, i.e.

$$\mathrm{supp}(\nu_t) \equiv X = \{x \in \mathbb{R}^n | x_i \geq -1,\ i = 1, \ldots, n\}$$

for all $t \in [0, T]$, then we get that

$$H_t \equiv \left\{ h \in \mathbb{R}^n \mid h_i \geq 0, \sum_{i=1}^{n} h_i \leq 1 \right\}$$

i.e. in order for V to stay positive the process h can take values in the n-dimensional unit simplex in $\mathbb{R}^n$.

Example 2: the 1-dimensional case

If $n = 1$ and $\mathrm{supp}(\nu_t) = [-m(t), M(t)]$, with $-1 \leq -m(t) \leq 0 \leq M(t) \leq +\infty$ for all $t \in [0, T]$, then

$$H_t = \left[-\frac{1}{M(t)}, \frac{1}{m(t)} \right]$$

for all $t \in [0, T]$, as in Liu *et al.* (2003).

Particular cases:

- only positive jumps: if $m(t) \equiv 0$, then the strategy h is unbounded from above;
- only negative jumps: if $M(t) \equiv 0$, then h is unbounded from below;
- unbounded jumps: if $-m(t) \equiv -1$ and $M(t) \equiv +\infty$, then $H_t = [0, 1]$, i.e. the inverstor will never take a leveraged or short position in the risky asset.

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Utility maximisation

We now fix a time horizon T in order to maximise, over the strategy h, the expected utility

$$\sup_h \mathbb{E}[U(V_T^h)] \tag{5}$$

where U is a CRRA utility function (e.g., $U(x) = \log x$ or $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$, $\gamma \neq 1$).

For technical reasons, we work with bounded strategies h (as seen, in many cases this is not a restriction): we assume that there exists a closed bounded convex set $H \subset \mathbb{R}^n$ such that $H \subset \mathrm{int}(\cap_{t\in[0,T]} H_t)$, and we call a strategy h *admissible* (notation $h \in \mathcal{A}[t,T]$), if it is predictable, $h_u \in H$ $\mathbb{P}$-a.s. for all $u \in [t,T]$ and Equation (3) has a unique strong solution $V^{t,v}$ for each initial condition $V_t = v$.

With this assumptions, V^h has finite variance for all $h \in \mathcal{A}[t,T]$.

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Dynamic programming

We define $J^h(t,v) := \mathbb{E}[U(V_T^{h;t,v})]$ and the *value function* as

$$J(t,v) := \sup_{h\in\mathcal{A}[t,T]} J^h(t,v) = \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}[U(V_T^{h;t,v})] \tag{6}$$

where $\{V_s^{t,v}, s \geq t\}$ is the solution of Equation (2) with initial condition $V_t := v > 0$, the second equality being justified by the Markovianity of V. The initial problem is thus equivalent to calculate $J(0, V_0)$.

It is well known that, by the dynamic programming principle, for all u such that $t+u \leq T$ we can write

$$\begin{aligned} J(t,v) &= \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}[\mathbb{E}[U(V_T^{h;t,v})|\mathcal{F}_{t+u}]] = \\ &= \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}\left[U\left(V_T^{h;t+u,V_{t+u}^{h;t,v}}\right)\right] = \\ &= \sup_{h\in\mathcal{A}[t,t+u]} \mathbb{E}[J(t+u, V_{t+u}^{h;t,v})] \end{aligned}$$

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The HJB equation

By formal arguments (Itô's formula on J, limit for $u \to 0$), we arrive to the HJB (Hamilton-Jacobi-Bellman) equation

$$\frac{\partial J}{\partial t}(t,v) + \sup_{h \in A} A^h J(t,v) = 0 \tag{7}$$

with

$$A^h J(t,v) = \frac{\partial J}{\partial v}(t,v) v \langle h, \mu(t) \rangle + \frac{1}{2} v^2 \langle \Sigma(t) h, h \rangle \frac{\partial^2 J}{\partial v^2}(t,v) + \\ + \int_{\mathbb{R}^n} \left(J(t, v + v\langle h, x \rangle) - J(t,v) - v \langle h, x \rangle \frac{\partial J}{\partial v}(t,v) \right) \nu_t(dx)$$

and terminal condition

$$J(T,v) = U(v) \tag{8}$$

The verification theorem

Theorem (Verification Theorem). Let K be a classical solution to the HJB equation with terminal condition $J(T,v) = U(v)$ and such that the Dynkyn formula

$$\mathbb{E}[f(T, V_T^h)] - \mathbb{E}[f(t, V_t^h)] = \mathbb{E}\left[\int_t^T A^{h_u} f(u, V_u^h)\, du\right]$$

holds for all $h \in \mathcal{A}[t,T]$. Then, for all $(t,v) \in [0,T] \times \mathbb{R}$:

(a) $K(t,v) \geq J^h(t,v)$ for every admissible control $h \in \mathcal{A}[t,T]$;

(b) if there exists an admissible control $h^* \in \mathcal{A}[t,T]$ such that

$$h_s^* \in \arg\max_h A^h K(s, V_s^h) \qquad \mathbb{P}\text{-a.s. for all } s \in [t,T],$$

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CRRA utility

We now consider a CRRA utility function, i.e. $U(v) = \frac{v^{1-\gamma}}{1-\gamma}$ with $\gamma > 0, \gamma \neq 1$ or $U(v) = \log v$ ("$\gamma = 1$"), and search for a solution of the kind $J(t,v) = U(e^{\varphi(t)}v)$, with $\varphi(t)$ deterministic function of time with terminal condition $\varphi(T) = 0$.

After some calculation (and dividing for $(ve^{\varphi(t)})^{1-\gamma}$) the HJB equation becomes

$$0 = \varphi'(t) + \sup_{h \in \mathcal{A}} F(t,h)$$

where $F(t,h)$ *does not depend on* v anymore!

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CRRA utility, cont.

The function F is defined for $\gamma \neq 1$ as

$$
\begin{aligned}
F(t,h) \quad := \quad & \langle h, \mu(t)\rangle - \frac{1}{2}\gamma\langle A(t)h, h\rangle + \\
& + \int_{\mathbb{R}^n} \left[\frac{1}{1-\gamma}\Big((1+\langle h, x\rangle)^{1-\gamma} - 1 \Big) - \langle h, x\rangle \right] \nu_t(dx)
\end{aligned}
$$

and for $\gamma = 1$ as

$$
F(t,h) := \langle h, \mu(t)\rangle - \frac{1}{2}\langle \Sigma(t)h, h\rangle + \int_{\mathbb{R}^n} \Big(\log(1+\langle h, x\rangle) - \langle h, x\rangle \Big) \nu_t(dx)
$$

The function F is strictly concave on h, as the sum of a linear function, a nonpositive quadratic form and a strictly concave function. Thus, being F strictly concave on the convex set H, there exists a unique solution $h^* \in H$.

CRRA utility: the optimal strategy

$\rightsquigarrow$ the optimal strategy $h^*(t)$ only depends on $\mu(t)$, $\sigma(t)$ and $\nu_t(dx)$. Thus, it is totally myopic, i.e. it does not depend on V or S^i, $i = 1, \ldots, n$, nor on the time to maturity $T - t$ (quite typical of CRRA utility functions with additive processes).

In the time-homogeneous case, i.e. when $\mu(t) \equiv \mu$, $\sigma(t) \equiv \sigma$ and $\nu_t \equiv \nu$, the optimal strategy h^* consists in investing wealth proportions in each risky asset which are constant in time.

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CRRA utility, concl.

Define now $\lambda(t) := F(t, h^*(t))$; then the HJB equation becomes

$$0 = \varphi'(t) + \lambda(t)$$

with terminal condition $\varphi(T) = 0$, and we have $\varphi(t) = \int_t^T \lambda(u)du$, and

$$J(t, v) = U(ve^{\varphi(t)})$$

After checking for the Dynkyn formula (integrability conditions), the verification theorem allows us to conclude that this is the value function of our optimization problem.

First-order condition

We now make the assumption that h^* satisfies the first-order condition

$$0 = \mu(t) - \gamma\Sigma(t)h^*(t) + \int_{\mathbb{R}^n} x((1 + \langle h^*(t), x\rangle)^{-\gamma} - 1)\ \nu_t(dx) \quad (9)$$

- The fact that the first-order condition (9) has a solution $h^*(t) \in H$ has to be verified case by case: see the examples at the end for numerical cases when this does NOT happen.
- Even proving that for all $t \in [0, T]$ the first-order condition (9) has a solution $h^* \in H_t$ (i.e. somewhere in the admissible values) does not seem an easy task: in fact, even in the time-homogeneous case (Kallsen 2000) this is assumed and not proved.
- In some particular cases, the existence of an optimal $h^*(t) \in H$ can be proved (Korn *et al.* 2003 for the logarithmic case, Callegaro-V. 2009 for the general case).

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- The fact that the first-order condition (9) has a solution $h^*(t) \in H$ has to be verified case by case: see the examples at the end for numerical cases when this does NOT happen.
- Even proving that for all $t \in [0, T]$ the first-order condition (9) has a solution $h^* \in H_t$ (i.e. somewhere in the admissible values) does not seem an easy task: in fact, even in the time-homogeneous case (Kallsen 2000) this is assumed and not proved.
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A decomposition for h_t^*

Now we compare our solution with the case when there are no jumps, i.e. $\nu_t \equiv 0$ for all $t \in [0, T]$. If the optimal portfolio proportions h^* satisfy (9) and the $\Sigma(t)$ are positive definite for all $t \in [0, T]$ (i.e., $d \geq n$ and the $\sigma(t)$, $t \in [0, T]$ have all full rank n), then

$$h_t^* = \frac{1}{\gamma}\Sigma^{-1}(t)(\mu(t) - \mu_t^J(h_t^*)) = \frac{1}{\gamma}\Sigma^{-1}(t)\mu(t) - \frac{1}{\gamma}\Sigma^{-1}(t)\mu_t^J(h_t^*)$$

where, for all $h \in H$, $t \in [0, T]$, $\mu_t^J(h)$ is the vector defined by

$$\mu_t^J(h) := \int_{\mathbb{R}^n} x(1 - (1 + \langle h, x \rangle)^{-\gamma}) \, \nu_t(dx)$$

which can be interpreted as a "jump dividend", i.e. a term which subtracts (if positive) something from the yield of the risky assets.

Analysis

In the general n-dimensional case, however, one cannot say if $\mu_t^J(h_t^*)$ has all positive components or not, and even in this case one has to take into account the fact that $\Sigma(t)$ can possibly be non-diagonal and transform positive vectors in vectors with some negative components.

The situation is different when $n = 1$: in this case we generalise a result of Framstad *et al.* (1998) and find out that the fraction of optimal portfolio invested in the risky asset in the presence of jumps is always less in absolute value than the corresponding fraction without jumps, and always with the same sign.

In mathematical terms, if $\sigma(t) > 0$ for all $t \in [0, T]$, then for all $t \in [0, T]$ one of the following holds:

$$0 \leq h_t^* \leq \frac{\mu(t)}{\gamma\sigma^2(t)} \quad \text{or} \quad \frac{\mu(t)}{\gamma\sigma^2(t)} \leq h_t^* \leq 0$$

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Second-order approximation

We can give another interpretation of this result, supported by strong numerical evidence, by making a first-order approximation in Equation (9):

$$0 = \mu(t) - \gamma\Sigma(t)\tilde{h} + \int_{\mathbb{R}^n} x(1 - \gamma\langle\tilde{h}, x\rangle + o(\gamma\langle\tilde{h}, x\rangle) - 1)\ \nu_t(dx)$$

By neglecting the term $o(\gamma\langle\tilde{h}, x\rangle)$, one arrives at

$$\mu(t) - \gamma\sum_{j=1}^{n} a_{ij}(t)\tilde{h}_j - \gamma\sum_{j=1}^{n}\int_{\mathbb{R}^n} x_i x_j \tilde{h}_j \nu_t(dx) = 0$$

We can now collect the vector $\tilde{h}$, which now appears linearly.

Second-order approximation, cont.

We obtain the approximation

$$\mu(t) - \gamma[\Sigma_t + C_t]\tilde{h} = 0 \tag{10}$$

where $C_t = (C_{ij}(t))_{ij}$ is the second moment matrix of the Levy measure ν_t, defined as

$$C_{ij}(t) := \int_{\mathbb{R}^n} x_i x_j \nu_t(dx)$$

(notice that $\Sigma_t + C_t$ is then the local covariance matrix of the n-dimensional driving process R). Finally one can think to approximate the optimal portfolio proportions with

$$h^*_t \simeq \tilde{h}_t := \frac{1}{\gamma}[\Sigma_t + C_t]^{-1}\mu(t) \tag{11}$$

i.e., the optimal portfolio proportions are near to the corresponding ones in the no-jump case when we substitute the volatility matrix Σ_t with the total covariance matrix $\Sigma_t + C_t$.

The Kou model

The Kou model is a jump-diffusion 1-dimensional model.

Levy density, ordinary exponential form ($S_t = e^{L_t}$):

$$\nu_L(dx) = \lambda(1-p)\eta_+ e^{-\eta_+ x} 1_{\{x>0\}}\, dx + \lambda p \eta_- e^{\eta_- x} 1_{\{x<0\}}\, dx$$

with $\eta_+ > 1$, $\eta_- > 0$ and $p \in [0,1]$.

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Numerical example with $\eta_+ = 10$, $\eta_- = 5$, $\lambda = 1$, $p = 0.4$, $\sigma = 0.16$ and $\mu = 0.0328$. Plug these values in Equation (9) for different values of γ from 0.2 to 2 and compare the results with the analogous optimal portfolios in the purely diffusive case, both with the original volatility and with the total variance:

γ	0.2	0.6	1 (log)	1.4	1.8
h^*	max H	0.83690	0.51773	0.37345	0.29183
h^*, no jumps	6.40625	2.13542	1.28125	0.91518	0.71181
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The Variance Gamma model (1 dim.)

The Variance Gamma model is an infinite activity model, which usually is used without a diffusion component.

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Usual interpretation: the Variance Gamma model is what results when the log-price follow a Brownian motion with drift, but with a discontinuous time change given by a so-called Gamma process G:

$$L_t := (\theta u + \sigma B_u)|_{u=G(t)}$$

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The Variance Gamma model, cont.

Since the Variance Gamma model has infinite variation, we do not need to add also a Brownian component. We can however make a comparison between this model and a geometric Brownian motion with the same first two moments.

Numerical example with $\lambda_+ = 39.78$, $\lambda_- = 20.26$, $c = 5.93$ and $\mu = 0.005$. This time we compare the results with the analogous optimal portfolios for a GBM with the same first two moments:

γ	0.2	0.6	1	1.4	1.8
h^*	$\max H$	$\max H$	0.77569	0.55825	0.43597
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A multidimensional example (Callegaro-V. 2009)

Assume that $\nu := \sum_{j=1}^m \lambda_j \delta_{c^j}$, with $\lambda_j > 0$, $c^j := (c_1^j, \dots, c_n^j) \in X$ and δ_x is the Dirac delta centered in x, i.e. the measure such that $\delta_x(B) = \mathbf{1}_B(x)$. This corresponds to N being the random Poisson measure corresponding to a multivariate Poisson process.

In this case, the first order conditions read

$$\begin{aligned} 0 &= \mu_i - \gamma \sum_{j=1}^n a_{ij} h_j + \int_{\mathbb{R}^n} \left[\left(1 + \langle h, x \rangle\right)^{-\gamma} x_i - x_i \right] \nu(dx) = \\ &= \mu_i - \gamma \sum_{j=1}^n a_{ij} h_j + \sum_{i=1}^m \lambda_j c_i^j \left[\left(1 + \langle h, c^j \rangle\right)^{-\gamma} - 1 \right] \quad \forall i = 1, \dots, n \end{aligned}$$

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