Dual Pricing of Swing Options with Bang-Bang Control

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Swing Options

- A kind of American-type derivative
- Traded in energy markets, such as gas and electricity
- Characteristics
  - Multiple rights
  - The availability of volume change
    - Option buyer and seller agree to trade energy in the future

- Difficult to price swing options analytically
  - Constraints for changing volume

➡ Pricing swing options numerically

Previous Works

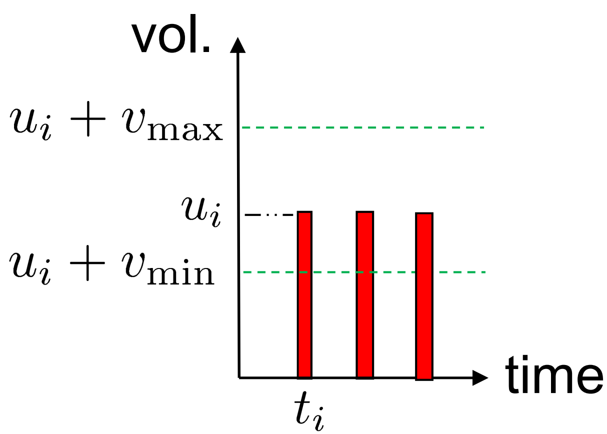
- Numerical methods for pricing American derivatives
  - lower bounds for the price
    - Least-squares Monte Carlo method
      (Longstaff and Schwartz (01))
    - Extension of LSM to swing options
      (Dörr (03), Barrera-Esteve et al. (06))
  - upper bounds for the price
    - Dual approach
      (Rogers (02), Haugh and Kogan (04), Bender (08), etc.)
- Applying the dual approach in the previous works to swing options is difficult
  - Swing options have flexibility relating to volume

My Contribution

- Extend a dual approach for pricing swing options with bang-bang control
  - Show monotonicity of the optimal exercise strategy
  - Introduce a second-order difference of the price
  - Decompose the pricing problem into single optimal stopping problems
  - Obtain an upper bound

Setup: Swing Options

- Underlying asset price process: $\{X_t\}$
- Possible exercise dates: $t_i \quad (i = 0, 1, \ldots, T)$
- Number of rights: $L \ (\leq T+1)$
- All rights must be exercised by t_T
- Bang-bang control
  - When a holder exercises a right, he changes traded volume from u_i to $\begin{cases} u_i + v_{\max} \ (v_{\min} \leq 0 \leq v_{\max}) \\ u_i + v_{\min} \end{cases}$



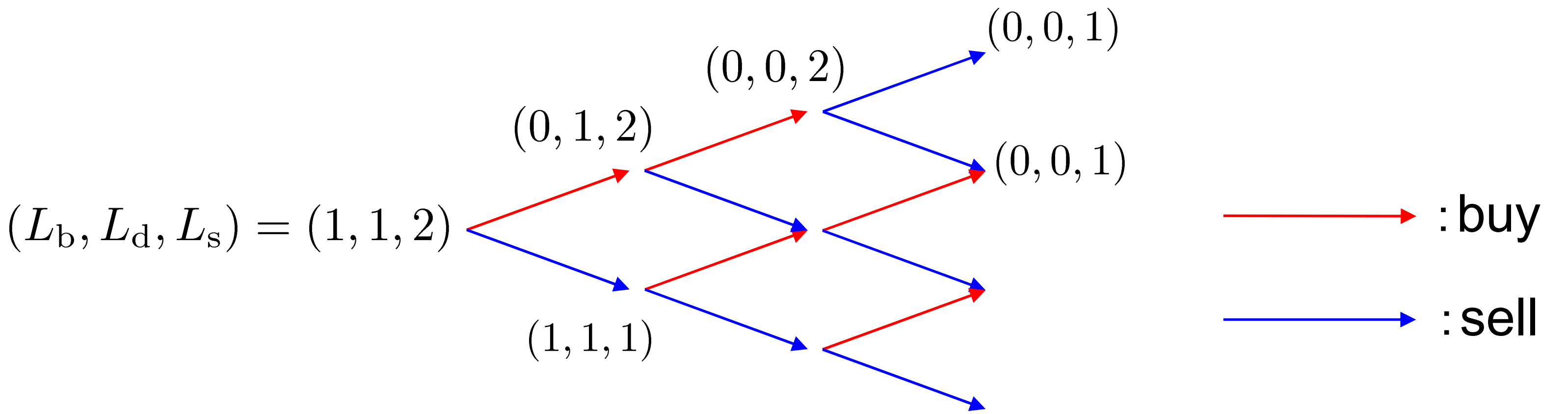
  - Numbers of choosing $u_i + v_{\max}$ and $u_i + v_{\min}$ are not less than L_{b} and L_{s} (buying) (selling)

Setup: Swing Options (cont'd)

- Constraints can be replaced by rights $(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$

  L_b : Number of obligations to buy
  L_d : Number of straddles
  L_s : Number of obligations to sell

- Transition tree for the number of rights



Formulation of Pricing

- Payoff $Z^{\mathrm{b}}(i) = v_{\max}(X_{t_i} - K)$ (K : strike price)
  $Z^{\mathrm{s}}(i) = v_{\min}(X_{t_i} - K)$

- Price of a swing option with $(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})$ at t_i

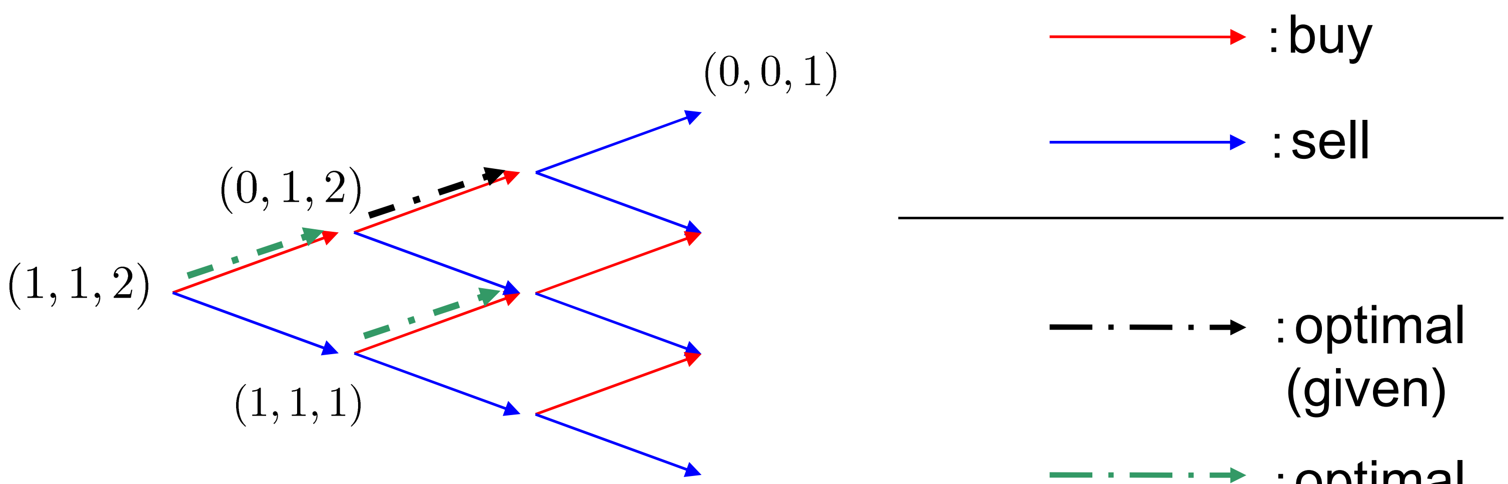
$$
\begin{aligned}
V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = \max [&\mathrm{E}_i[V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)],\\
&Z^{\mathrm{b}}(i) + \mathrm{E}_i[V(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)],\\
&Z^{\mathrm{s}}(i) + \mathrm{E}_i[V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i+1)]]
\end{aligned}
$$

- Optimal strategy on $(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})$

$$
\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = \begin{cases}
\text{“Non-exercise”} & (V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = \mathrm{E}_i[V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)])\\
\text{“buy”} & (V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = Z^{\mathrm{b}}(i) + \mathrm{E}_i[V(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)])\\
\text{“sell”} & (V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = Z^{\mathrm{s}}(i) + \mathrm{E}_i[V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i+1)])
\end{cases}
$$

Monotonicity for Swing Options

- Optimal strategies between different rights hold monotonicity
- Example on a transition tree



- If $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i) =$ "buy", then $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) =$ "buy"
- If $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) =$ "buy", then $\xi(L_\mathrm{b}+1, L_\mathrm{d}, L_\mathrm{s}-1, i) =$ "buy"

Dual Approach

- Proposed by Rogers (02) and Haugh and Kogan (04)
- American options price $V(0)$ satisfies

$$V(0) = \sup_{0 \le \tau \le T} \mathrm{E}[Z(\tau)]$$

$$\le \mathrm{E}[\max_{i=0,\cdots,T} (Z(i) - M(i))] \qquad (Z(i) : \text{payoff})$$

for any martingale $M(i)$ $(M(0) = 0)$

- Equality holds for the martingale part $M^*(i)$ of the Doob decomposition of $V(i)$
- An upper bound for the true price can be calculated

Difficulty in Extension for Swing Option

- Multiple rights
  - Natural approach: decomposition into options with a single decision
  - For multiple American options, some studies evaluate a difference for the number of rights

$$\Delta V(L,0) = V(L,0) - V(L-1,0) \qquad L: \text{number of rights}$$

- Possibility of choice to buy or sell
  - $\Delta V(L,0)$ does not reflect the choice $\longrightarrow$ unnatural

How do we decompose?

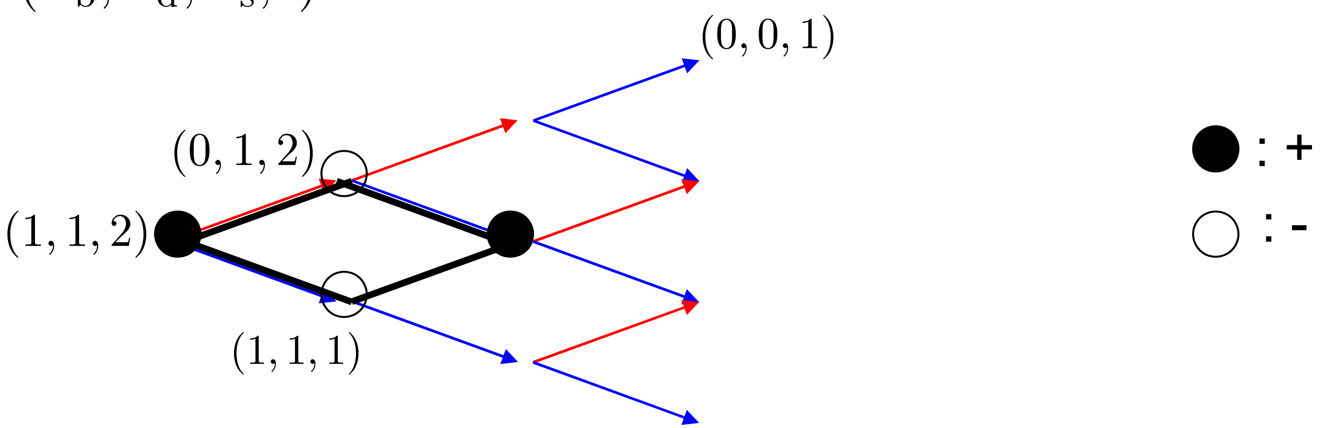
Introducing Second-Order Difference

- Second-order difference for the number of rights

$$\Delta\Delta V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) \equiv V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) - V(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i)$$
$$-V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i) + V(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}-1, i)$$

- Price of the swing option can be decomposed into $\Delta\Delta V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i)$

Example:



$$V(1,1,2,i) = \sum_{l \in \mathcal{L}(1,1,2)} \Delta\Delta V(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i)$$

$\mathcal{L}(1,1,2)$: Node set

l : Abbreviation of $(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s})$

Introducing Second-Order Difference

- Second-order difference for the number of rights

$$\Delta\Delta V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \equiv V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) - V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) - V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) + V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i)$$

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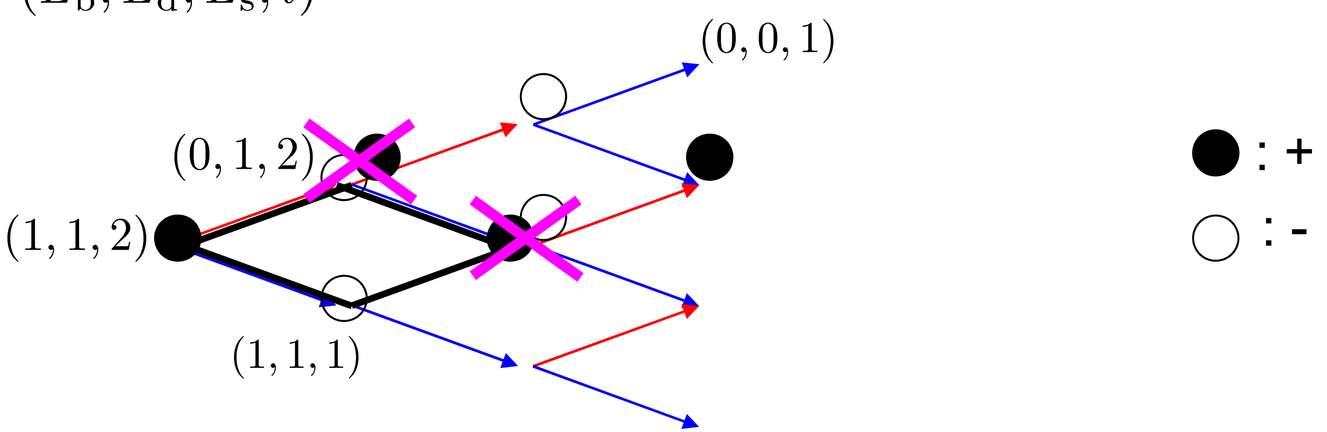
Example:



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Main Result

- Consider optimal stopping problems that correspond to second-order differences $\Delta\Delta V(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i)$

Theorem:

If an exercise strategy ξ is monotone and a good estimator of the optimal strategy, then it holds that

$$V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, 0) = \sum_{l \in \mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})} \Delta\Delta V(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, 0)$$
$$\leq \sum_{l \in \mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})} \sup_{0 \leq \tau \leq T} \mathrm{E}[Z_l^{\xi}(\tau)]$$

$Z_l^{\xi}(i)$: adjusted payoff (discuss later)

- Equality holds for the optimal exercise strategy ξ^*

Main Result (cont'd)

Theorem:

For any martingale $M_l(i)$, it holds that

$$V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, 0) = \sum_{l \in \mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})} \sup_{0 \le \tau \le T} \mathrm{E}[Z_l^{\xi^*}(\tau)]$$

$$\le \sum_{l \in \mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})} \mathrm{E}\left[\max_{i=0,\ldots,T} [Z_l^{\xi^*}(i) - M_l(i)] \right]$$

- Equality holds for the martingale part $M_l^*(i)$ of the Doob decomposition of $\Delta\Delta V(\hat{l}_{\mathrm{b}}^{\xi^*}(i), \hat{l}_{\mathrm{d}}^{\xi^*}(i), \hat{l}_{\mathrm{s}}^{\xi^*}(i), i)$

$\hat{l}_{\mathrm{b}}^{\xi}(i), \hat{l}_{\mathrm{d}}^{\xi}(i), \hat{l}_{\mathrm{s}}^{\xi}(i)$: number of residual rights determined by ξ

Concept of $Z_l^{\xi}(i)$

- For example, consider Z at t_0
- Depend on the number of exercised terms in the second-order difference by ξ

Case1. not more than one term

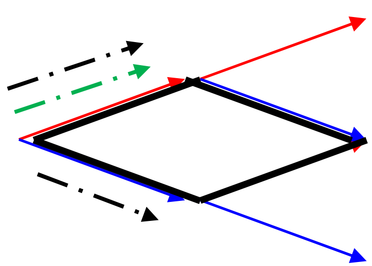
$$Z_l^{\xi}(0) = \max$$ [payoff from buying, payoff from selling]

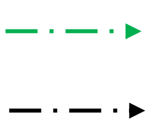
Case2. two terms

$$Z_l^{\xi}(0) = 0$$

Case3. not less than three terms

$$Z_l^{\xi}(0) = -\infty$$





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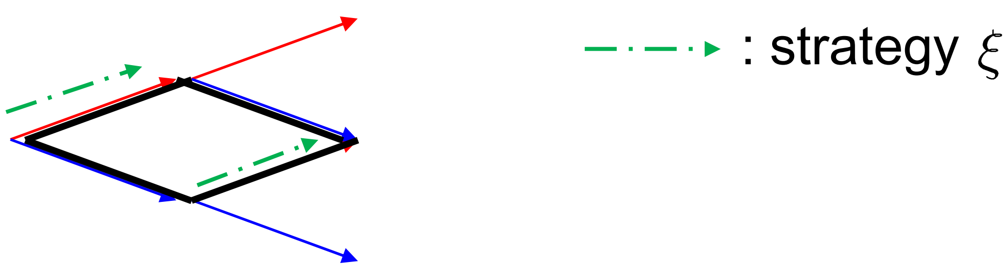

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Numerical Example

- Asset price process: mean-reverting process

$$\mathrm{d}X_t = -3 \cdot (X_t - 40)\mathrm{d}t + 0.5\mathrm{d}W_t,\ \ X_{t_0} = 40$$

- Strike price $K = 40$
  Maturity $T = 20$ and 100
  $(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$: $(2,2,2),\ (6,6,6)$ and $(10,10,10)$

Prop.

For a mean-reverting process, the optimal exercise boundary is determined

Numerical Example: Algorithm

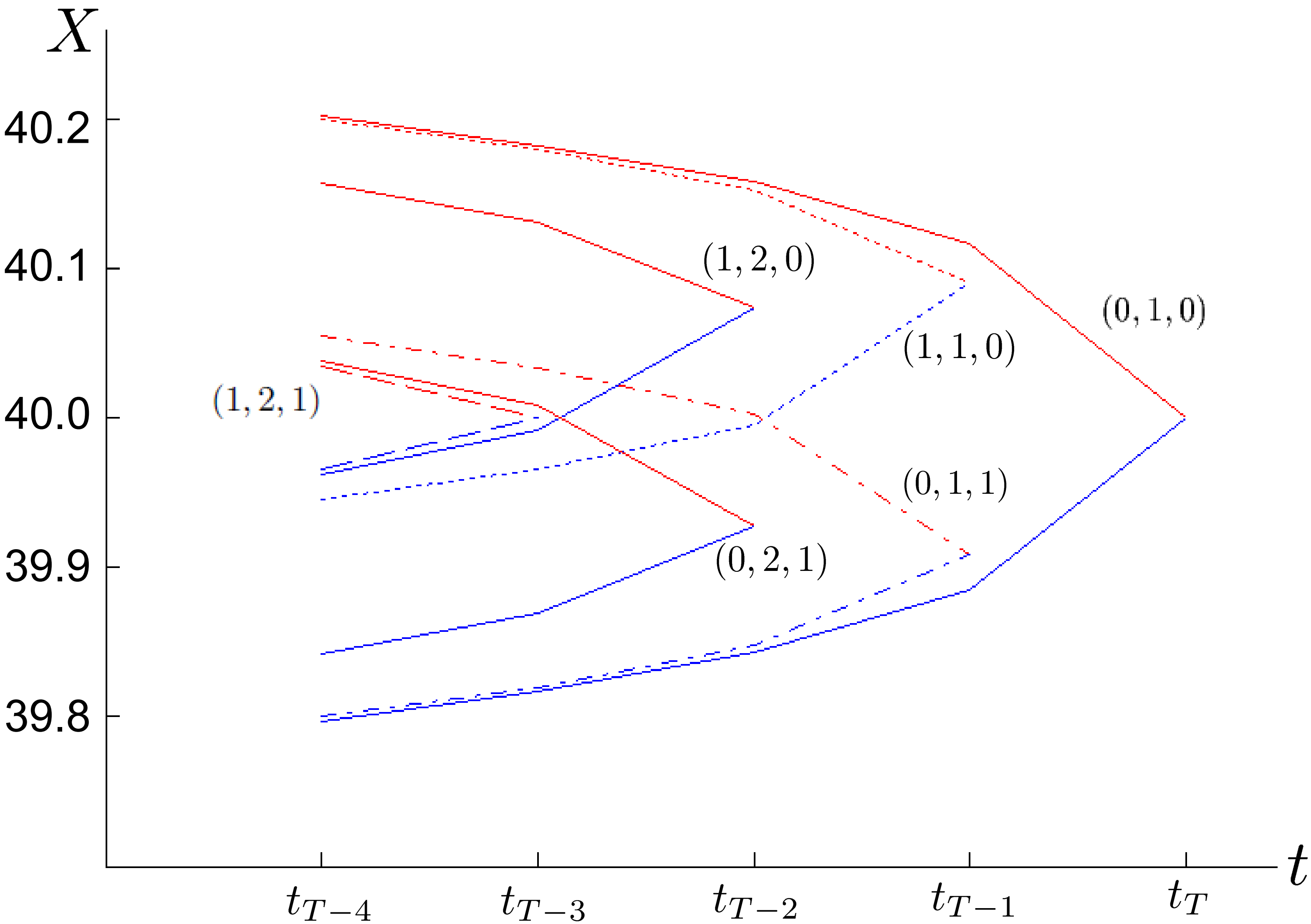
- Based on Andersen and Broadie (04) and Bender (08)

Step 1. The least-squares Monte Carlo regression

Step 2. Estimating optimal exercise boundary
– using coefficients obtained from Step.1

Step 3. Estimating martingales in the theorem
– from estimated exercise boundary

Numerical Result: Exercise Boundary



Numerical Result: Price

Rights	$T = 20$		$T = 100$	
	lower	upper	lower	upper
$(2, 2, 2)$	0.8985	0.9007	1.9408	1.9426
	(0.0011)	(0.0006)	(0.0013)	(0.0015)
$(6, 6, 6)$	2.0638	2.0692	5.2251	5.2421
	(0.0024)	(0.0015)	(0.0033)	(0.0094)
$(10, 10, 10)$	-	-	7.9007	7.9364
	-	-	(0.0048)	(0.0079)

(Standard errors are in parentheses)

- Differences between upper and lower bounds are less than 1% of the price in all cases

Summary and Future Works

- For the swing option with bang-bang control,
  - the optimal strategy is monotone
  - the sum of optimal stopping problems corresponding to second-order differences gives an upper bound of the price

- Future works: extension for more complicated options
  - For constant daily and annual constraints, I will be able to extend in a similar way
  - For more general constraints, the dual problem for volume will give an upper bound