In Arrear Term Structure Products: No Arbitrage Pricing Bounds and The Convexity Adjustments

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In-arrear interest rate products

- Convexity adjustment (correction) occurs when the interest rate pays out at the wrong time and/or in the wrong currency..
- In-arrear swaps and in-arrear caps and floors differ from vanilla products in the floating leg
  - ▷ in the vanilla products, the payoff at each time t_i is based on the LIBOR rate observed at the date before, i.e. at t_{i-1}..
  - ▷ in the in-arrear products, the payoff at each time t_i is based on the LIBOR rate at t_i..
- Compared to the vanilla products, there is mismatch in cash flow timing

Valuation of in-arrear interest rate products

- Mostly when no further assumptions concerning the term structure of the interest rate are made, it is impossible to price these products
- Even when a specific term structure model is assumed, closed-form solutions can only be achieved in very few cases (e.g. LIBOR market model)
- Approximation methods are frequently used
  - the price of an in-arrear swap is usually approximated by the sum of a forward-starting vanilla swap and a convexity adjustment term
  - Similarly, the price of an in-arrear cap/floor can be decomposed into a vanilla cap/floor plus a convexity adjustment

Notations

- Consider an in-arrear payer swap
  - ▷ Assume that the variable interest payments (floating leg) are received at a set of equidistant time points in

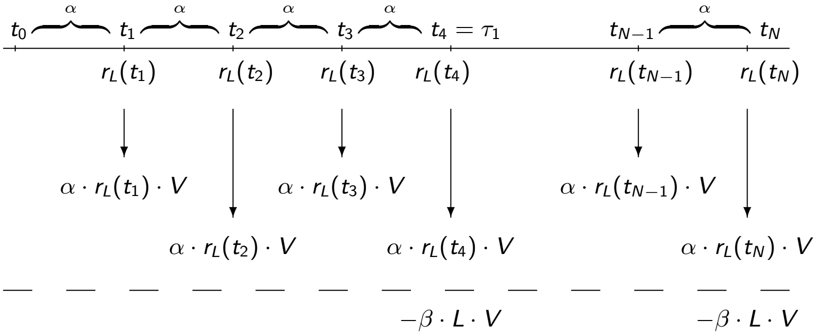
$$\underline{\underline{T}} = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\}$$

with $\alpha := t_{i+1} - t_i$.
  - ▷ The fixed leg pays fixed interest rate payments at a set of equidistant time points in

$$\Theta = \{0 = \tau_0 < \tau_1 < \cdots \tau_n := T\}$$

  - ▷ Let $\beta := \tau_{j+1} - \tau_j$. Assume that β is a multiple of α, i.e. $\beta = m \cdot \alpha,\ m \in \mathbb{N}$. It implies that $\tau_n = t_{n \cdot m} = t_N = T$.

Payoff structure of an in-arrear payer swap



Payoff structure of an in-arrear payer swap with $m = 4$ and $r_L(t_i, t_i, \alpha) := r_L(t_i)$

Accumulation of the variable interest rate from t_i to t_{i+1}

t_i $\qquad\qquad$ t_{i+1}

$\alpha \cdot r_L(t_i) \cdot V$

Accumulation

$(1 + \alpha \cdot r_L(t_i))$

$$\alpha \cdot r_L(t_i) \cdot V \cdot (1 + \alpha \cdot r_L(t_i))$$

$$= \underbrace{\alpha \cdot r_L(t_i) \cdot V}_{\text{variable interest payment of a vanilla swap}} + \underbrace{\alpha^2 \cdot r_L^2(t_i) \cdot V}_{\text{additional interest payments}}$$

Floating leg

- The initial arbitrage-free value of the floating leg of the in-arrear swaps can be decomposed into two parts:

  a) the initial arbitrage-free value of the floating leg of a forward-starting vanilla swap which starts to receive $\alpha \cdot V \cdot r_L(t_1)$ at time t_2 and ends with the last payment $\alpha \cdot V \cdot r_L(t_N)$ at time $t_{N+1} := t_N + \alpha$,

  b) and the initial arbitrage-free value of a sequence of additional variable interest payments:

$$\alpha^2 \cdot r_L^2(t_i) \cdot V, \quad i = 2, \ldots, N+1.$$

Arbitrage value: part a) of floating leg

- The payoff of the forward-starting vanilla swap can be duplicated by the following simple strategy:
  - ▷ At time 0 buy V zero coupon bonds with maturity t_1.
  - ▷ At time t_1, invest V as a rollover deposit with the time-varying LIBOR rate.
  - ▷ At the end t_{i+1} of each period $[t_i, t_{i+1}]$, $i = 1, \cdots, N$, keep the interest rate payment, and invest V further for one more period.
  - ▷ Repeat this process until $t_{N+1} := t_N + \alpha$. The terminal strategy value is $V + \alpha \cdot V \cdot r_L(t_N)$.
  - ▷ At time 0 sell V zero coupon bonds with maturity t_{N+1}.
- The initial arbitrage-free value of part a) of the floating leg corresponds to the value of the entire strategy, i.e. $VB(t_0, t_1) - VB(t_0, t_{N+1})$

Additional variable interest rate payments

- Impossible to determine the arbitrage-free values of this part without further assumptions about the term structure of the interest rate
- One trick is to make the following approximation

$$r_L(t_i, t_i, \alpha) := r_L(t_i) \approx r_L(t_0, t_i, \alpha)$$

  - ▷ The technique of replacing the random spot LIBOR rate by the forward LIBOR rate at time zero is the so-called convexity adjustments approach

Convexity adjustment for the initial arbitrage-free value

- The model-independent approximation for the initial arbitrage-free value of the in-arrear payer swap:

$$\begin{aligned}
&\text{payer-swap}_{ar}[r_L, L, V, \underline{\underline{T}}, \Theta] \\
\approx\, & V \cdot \left(B(t_0, t_1) - B(t_0, t_{N+1}) - \left(\sum_{j=1}^{n} \beta \cdot L \cdot B(\tau_0, \tau_j) \right) \right) \\
& + V \cdot \sum_{i=1}^{N} \Big(\underbrace{\frac{B(t_0, t_i) - B(t_0, t_{i+1})}{B(t_0, t_{i+1})}}_{\alpha r_L(t_0, t_i, \alpha)} \Big)^2 .
\end{aligned}$$

Assumption

- Assumption:

Assume an arbitrage-free financial market with a set of equivalent martingale measures $\mathbf{P}$. It is furthermore assumed that there exists at least an equivalent martingale measure $P^* \in \mathbf{P}$ such that P^* can be changed to a forward measure $Q^{\tau+\alpha}$ for each compounding period α and each time τ through the change-of-measure technique and under $Q^{\tau+\alpha}$ the forward LIBOR rate process $\{r_L(t,\tau,\alpha)\}_{t\leq\tau}$ is a martingale. It holds particularly

$$r_L(t_0,\tau,\alpha) = E_{Q^{\tau+\alpha}}[r_L(t,\tau,\alpha)].$$

Proposition

In any arbitrage-free interest rate model with the above assumption, the convexity adjustments of an in-arrear payer swap is a lower bound for the arbitrage-free price of an in-arrear payer swap:

$$\text{payer-swap}_{ar}[r_L, L, V, \underline{\underline{T}}, \Theta] \geq A(L)$$

where $\underline{\underline{T}} = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\}$ with $\alpha := t_{i+1} - t_i$ and $\Theta = \{0 = \tau_0 < \tau_1 < \cdots \tau_n : T\}$ with $\beta = \tau_{j+1} - \tau_j = m \cdot \alpha$. The pricing bound $A(L)$ owns the form of

$$A(L) = V \cdot B(t_0, t_1) - V \cdot B(t_0, t_{N+1}) + \sum_{i=1}^{N} V \cdot B(t_0, t_{i+1}) \left(\frac{B(t_0, t_i)}{B(t_0, t_{i+1})} - 1 \right)^2 - \sum_{j=1}^{n} \beta \cdot L \cdot V \cdot B(t_0, t_{j \cdot m}).$$

In particular, L^* with $A(L^*) = 0$ is a lower bound for the swap yield of an in-arrear payer swap.

Proof Sketch

What needs to be approximated in the in-arrear swap is the sequence of additional variable interest rate payments $\alpha^2 \cdot r_L^2(t_j, t_j, \alpha)V$, $j = 1, \cdots, N$:

$$\begin{aligned}
& E_{P^*}\left[e^{-\int_0^{t_{j+1}} r_u du}\alpha^2 r_L^2(t_j, t_j, \alpha)V \middle| \mathcal{F}_{t_0}\right] \\
=& B(t_0, t_{j+1})\alpha^2 V E_{Q^{t_{j+1}}}\left[r_L^2(t_j, t_j, \alpha)|\mathcal{F}_{t_0}\right] \\
=& B(t_0, t_{j+1})\alpha^2 V\left[\left(E_{Q^{t_{j+1}}}\left[r_L(t_j, t_j, \alpha)|\mathcal{F}_{t_0}\right]\right)^2 + Var_{Q^{t_{j+1}}}\left[r_L(t_j, t_j, \alpha)|\mathcal{F}_{t_0}\right]\right] \\
\geq& B(t_0, t_{j+1})\alpha^2 V\left(E_{Q^{t_{j+1}}}\left[r_L(t_j, t_j, \alpha)|\mathcal{F}_{t_0}\right]\right)^2 \\
=& B(t_0, t_{j+1})\alpha^2 V r_L^2(t_0, t_j, \alpha) \\
=& B(t_0, t_{j+1})V\left(\frac{B(t_0, t_j)}{B(t_0, t_{j+1})} - 1\right)^2.
\end{aligned}$$

Numbers

- Normal term structure of the interest rate, particularly that the log-rate of return of the zero coupon bonds are linearly increasing in time to maturity:

  $$y(t_0, t_i) = 2.5\% + 0.2\% \cdot (t_i - t_0).$$

  Consequently, zero coupon bonds have the value of

  $$B(t_0, t_i) = \exp\{-y(t_0, t_i)(t_i - t_0)\}.$$

  Other parameters are chosen as follows:

  $$\alpha = \frac{1}{4},\ \beta = \frac{1}{2},\ L = 3.5\%,\ V = 1$$

- We adopt the following function (c.f. Brigo and Mercurio (2001))

  $$\gamma(t, t_i) = [g + a(t_i - t)] \exp\{-b(t_i - t)\} + c$$

  with $a = 0.19085664$; $b = 0.97462314$; $c = 0.08089168$; $g = 0.01344948$

Numerical results

Maturity T in years	In-Arrear Payer Swap			
	LIBOR model		Convexity correction	
	Arb-free price	imp. L^*	Arb-free price	imp. L^*
1	-0.66764	2.81889	-0.66787	2.81866
2	-0.92879	3.01913	-0.92989	3.01856
3	-0.80648	3.21712	-0.80920	3.21616
4	-0.32717	3.41243	-0.33229	3.41106
5	0.47992	3.60467	0.47161	3.60286
6	1.58337	3.79346	1.57105	3.79117
7	2.95022	3.97841	2.93301	3.97562
8	4.54652	4.15918	4.52352	4.15584
9	6.33794	4.33540	6.30820	4.33148
10	8.29032	4.50675	8.25288	4.50221

Arbitrage-free prices for (in-arrear) payer swaps and implied swap yield based on exact pricing formulae and the corresponding convexity adjustments (pricing bounds).

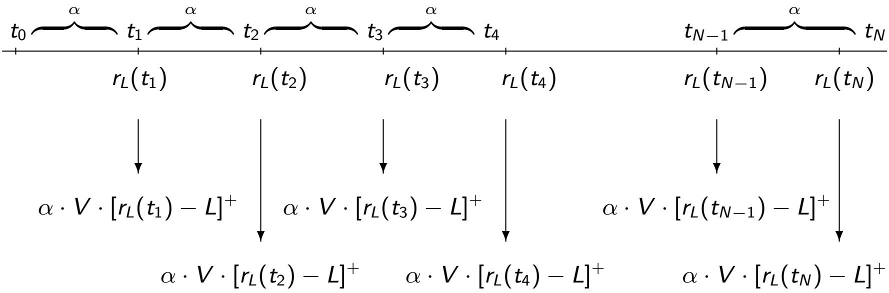
Notations

- As the reference payment time, we fix again an equidistant sequence of time points $\underline{\underline{T}} = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\}$ with $\alpha := t_{i+1} - t_i$, and L is now the cap/floor rate.
- In a vanilla cap, the contract holder receives $\alpha \cdot V \cdot [r_L(t_i) - L]^+$ at t_{i+1}, $i = 1, \cdots, N$. Hereby we have used $[x]^+ := \max\{x, 0\}$.
- The contract holder of an in-arrear cap receives

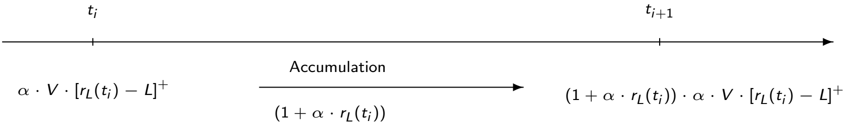
$$\alpha \cdot V \cdot [r_L(t_i) - L]^+$$

already at time t_i, $i = 1, \cdots N$.

Payoff structure of an in-arrear cap



Accumulation of the payment from t_i to t_{i+1}



Approximation

- Apparently, without further assumptions about the term structure of the interest rate, it is impossible to calculate the initial arbitrage-free value of the in-arrear cap.
- Use the initial forward LIBOR rate $r_L(t_0, t_i, \alpha)$ to approximate $r_L(t_i)$:

$$(1 + \alpha \cdot r_L(t_i)) \approx (1 + \alpha \cdot r_L(t_0, t_i, \alpha)) = \frac{B(t_0, t_i)}{B(t_0, t_{i+1})}.$$

- Approximation for the initial market value of the in-arrear cap:

$$Cap_{ar}[r_L, L, V, \underline{\underline{T}}, t_0] \approx \sum_{i=1}^{N} \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} \cdot \text{Caplet}(r_L, L, V, t_0, t_{i+1})$$

where $\text{Caplet}(r_L, L, V, t_0, t_{i+1})$ gives the initial arbitrage-free value of $\alpha \cdot V \cdot [r_L(t_i) - L]^+$ which becomes due at time t_{i+1}.

Proposition

In an arbitrage-free interest rate model with the above mentioned assumption, there exists a model-independent lower bound for the arbitrage-free price of an in-arrear cap, i.e.

$$Cap_{ar}[r_L, L, V, \underline{\underline{T}}, t_0] \geq \sum_{i=1}^{N} \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} \cdot \text{Caplet}(r_L, L, V, t_0, t_{i+1})$$

where $\underline{\underline{T}} = \{0 = t_0 < t_1 < t_2 < \cdots < t_N := T\}$ with $\alpha := t_{i+1} - t_i$ and Caplet is defined by

$$\text{Caplet}(r_L, L, V, t_0, t_{i+1}) := B(t_0, t_{i+1})\, E_{Q^{t_{i+1}}}[\alpha \cdot V \cdot [r_L(t_i) - L]^+ | \mathcal{F}_{t_0}].$$

Proof sketch

$$
\begin{aligned}
& Cap_{ar}[r_L, L, V, \underline{\underline{T}}, t_0] \\
=& \sum_{i=1}^{N} E_{P^*}\left[e^{-\int_0^{t_{i+1}} r_u du} (1 + \alpha \cdot r_L(t_i)) \cdot \alpha \cdot V \cdot [r_L(t_i) - L]^+ \Big| \mathcal{F}_{t_0} \right] \\
=& \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}}\left[(1 + \alpha \cdot r_L(t_i)) \cdot \alpha \cdot V \cdot [r_L(t_i) - L]^+ \Big| \mathcal{F}_{t_0} \right] \\
=& \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}}\left[\alpha \cdot V \cdot [r_L(t_i) - L]^+ \Big| \mathcal{F}_{t_0} \right] \\
& + \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}}\left[\alpha^2 \cdot V \cdot r_L(t_i)) \cdot [r_L(t_i) - L]^+ \Big| \mathcal{F}_{t_0} \right] \\
=& \sum_{i=1}^{N} \text{Caplet}(r_L, L, V, t_0, t_{i+1}) + \sum_{i=1}^{N} B(t_0, t_{i+1}) \cdot E_{Q^{t_{i+1}}}\left[\alpha^2 \cdot V \cdot r_L(t_i) \cdot [r_L(t_i) - L]^+ \Big| \mathcal{F}_{t_0} \right]
\end{aligned}
$$

...Proof sketch

The second term on the right-hand side can be rewritten as

$$\sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[\alpha^2 \cdot V \cdot r_L(t_i) \cdot [r_L(t_i) - L]^+ \middle| \mathcal{F}_{t_0} \right]$$

$$= \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[[r_L(t_i) - L]^+ \middle| \mathcal{F}_{t_0} \right] \cdot E_{Q^{t_{j+1}}} [r_L(t_i) | \mathcal{F}_{t_0}]$$

$$+ \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) Cov_{Q^{t_{i+1}}} \left[r_L(t_i), [r_L(t_i) - L]^+ \middle| \mathcal{F}_{t_0} \right]$$

$$\geq \alpha^2 \cdot V \cdot \sum_{i=1}^{N} B(t_0, t_{i+1}) E_{Q^{t_{i+1}}} \left[[r_L(t_i) - L]^+ \middle| \mathcal{F}_{t_0} \right] \cdot E_{Q^{t_{j+1}}} [r_L(t_i) | \mathcal{F}_{t_0}]$$

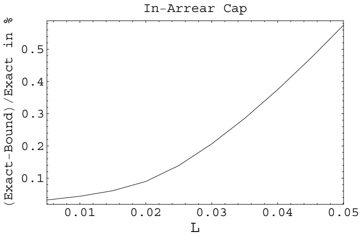
$$= \sum_{i=1}^{N} \alpha \cdot r_L(t_0, t_i, \alpha) \cdot B(t_0, t_{i+1}) \cdot \alpha \cdot V E_{Q^{t_{i+1}}} \left[[r_L(t_i) - L]^+ \middle| \mathcal{F}_{t_0} \right]$$

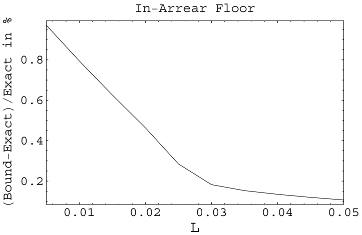
Numerical results

Maturity T in years	Cap contract			Floor contract		
	vanilla	in-arrear	convexity adjustment	vanilla	in-arrear	convexity adjustment
1	0.00452	0.00456	0.00455	0.77786	0.78289	0.78309
2	0.10260	0.10370	0.10341	1.24444	1.25255	1.25329
3	0.40505	0.40976	0.40854	1.53388	1.54391	1.54526
4	0.95353	0.96536	0.96241	1.71438	1.72562	1.72752
5	1.75991	1.78306	1.77752	1.82707	1.83908	1.84142
6	2.81902	2.85820	2.84917	1.89815	1.91065	1.91335
7	4.11421	4.17446	4.16100	1.94385	1.95666	1.95963
8	5.62150	5.70801	5.68917	1.97392	1.98695	1.99012
9	7.31264	7.43066	7.40545	1.99423	2.00739	2.01072
10	9.15733	9.31198	9.27944	2.00828	2.02155	2.02500

Arbitrage-free prices for (in-arrear) caps and floors based on exact pricing formulae and the corresponding convexity adjustments (pricing bounds).

Comparison of the convexity adjustment and the exact value





|Exact value - Convexity adjustment|/Exact value in percentage with parameters: $\alpha = \frac{1}{4},\ T = 5$.

Conclusion

- In the present paper, we provide a strong theoretical argument to support the convexity adjustments approach, a rule of thumb used by practitioners to value in-arrear products.
- Our results exclusively depend on the no-arbitrage condition which ensures the existence of certain forward risk adjustment measures. They are model-independent and in effect give pricing bounds for in-arrear term structure products.