An Equilibrium Model for Term Structure of Interest Rates with Consumption Commitments

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6th World Congress of Bachelier Finance Soc.

June 22-27, 2010

Sidney Homer

"Free market long-term rates of interest for any industrial nation, properly charted, provide a sort of fever chart of the economic and political health of that nation."

Yield of a bond

is the annual continuous compound return rate. For a (zero coupon) bond with term i, current price p and pays one unit at maturity, denote the yield by $y(i)$.We have $pe^{y(i)i} = 1$ or

$$y(i) = \frac{-\ln(p)}{i}.$$

Term structure

refers to the function relationship $i \longrightarrow y(i)$. It is often graphically represented by graph(y) – a yield curve.

Existing term structure models

- Short rate model.
- No arbitrage model.

The short rate model

- Assuming a model for the short period bonds in to the future, derive term structures according to principle of consistency–no arbitrage opportunities.
- O. Vasicek pioneered this model in 1977.
- Many generalizations afterwards: Cox-Ingersoll-Ross (1985), Chan-Karolyi-Longerstuff-Sanders(1992), Diffie-Kan (1996).
- Simplicity is a tradition in this line of research.

No arbitrage model

- Using the current yield curve as a starting point, derive term structures according to principle of no arbitrage.
- Ho and Lee (1986) and Hull and White (1990) are representatives of early works in this direction.
- Heath-Jarrow-Merton (1992) provided a general theoretical framework.

Our approach: Motivation

to reflect the impact of consumption commitment. Examples:

- Saving for education and retirement.
- Insurance companies projected need of funds due to their insurance portfolio.
- Foreign countries demand commeasurable with their structure needs.

General equilibrium model

- Since our focus is on the supply and demand in nature, a general equilibrium model becomes a natural choice.
- We use a model that develops the influential Lucas (1978) model for stock pricing.
- Comparing to the existing models, this model is more general in principle
- but, in this initial investigation, less sophisticated technically.

Results

- Establish dynamic programming equations
- Existence and uniqueness of the equilibrium term structures.
- Computation algorithms.
- Sample testing results that are consistent with intuition and qualitative policy ramifications.

Related extension of the Lucas model

Model of Judd, Kubler and Schmedders (2003)

- Also involve both stock and bonds.
- Focuses on Perato inefficiency of the equilibrium, portfolio separation and trading volume.
- Involving multiple agents – generalizes the Lucas model in a different direction.
- Our model use only one representative agent as Lucas and is computationally tractable.

Model: The producer

The producer is the aggregate of all agents producing consumption goods. We represent it as one unit share of stock whose products in each period can be regarded as a dividend process D_t. We assume that D_t is a stationary markovian process defined by its transition function

$$F(D', D) = \mathrm{Prob}(D_{t+1} \leq D' \mid D_t = D). \tag{1}$$

The ex-dividend price at t of the stock is denoted by s_t. D_t is the sole source of uncertainty in this model.

Assumption on the dividend process

We make the following technical assumption on $D = D_t$.

(A1) The dividend D_t has a compact positive range $[D_m, D_M]$ and that the mapping $T : C([D_m, D_M], R^m) \to C([D_m, D_M], R^m)$ defined by

$$(Tg)(D) = \int g(D')dF(D', D)$$

maps any bounded set in $C([D_m, D_M], R^m)$ to an equi-continuous subset of $C([D_m, D_M], R^m)$.

Condition (A1) is automatically satisfied when D has only a finite number of states.

Model:The government

fund its operation through taxation qD_t and issuing (zero coupon) bond with duration 1 through m periods. The vector

$$p_t(D_t) = (p_t(1, D_t), \ldots, p_t(m, D_t))^\top$$

represents the prices of the m bonds. We will use vector

$$B_t = (B_t(1), \ldots, B_t(m))^\top$$

to denote the quantity of outstanding bonds at t.
These bonds will be purchased by the consumer and acting as a medium for transferring consumption power.

Assumption on par value of the bonds

The total par value of the government bonds is assumed to be a fixed portion of the total expected production, that is

$$\vec{1} \cdot B_t = \sum_{i=1}^{m} B_t(i) = \alpha E[D_t] = \alpha E[D], \tag{2}$$

where α reflects the government fiscal policy towards debt. We can roughly think α as a proxy for the ratio of total government debt to GDP.

Model: The consumer

wishes to maximize

$$E\left[\sum_{t=0}^{\infty}\beta^t\left(u(c_t)+\phi(a_t-\gamma_t)\right)\right]. \tag{3}$$

c_t – stochastic processes of consumption.

a_t – allocation for (partially) satisfying the commitment.

β – discount factor.

u – concave utility function

ϕ – concave penalty function for deviation from commitments.

w_t – net worth at time t.

γ_t – the consumption commitment at period t.

Consumer's portfolio

The consumer distributes his consumption power among different time periods by holding and adjusting a portfolio

$$(\theta_t, \psi_t) = (\theta_t, \psi_t(1), \dots, \psi_t(m))$$

in which θ_t represents the the share of stocks and $\psi_t(i), i = 1, 2, ..., m$ represents the share of the i period bond. In general, c_t, a_t, θ_t and ψ_t are all depend on D_t.

Assumptions on the penalty function

The penalty function $\phi(s)$ attains its maximum at $s = 0$ and $\phi(0) = 0$. Insufficient allocation to meet the commitment indicated by $a_t - \gamma_t < 0$ will be penalized.

Assuming ϕ to be concave means that as the deficit in satisfying the consumption commitment $a_t - \gamma_t$ grows larger, the rate of marginal penalty will increase. With this model there is no incentive to allocate a_t above the consumption commitment γ_t because doing so will reduce the utility and increase the penalty (reduce the value of $\phi(a_t - \gamma_t)$).

When $\gamma_t = 0$ this means that we will allow a_t to take negative value.

Model: Constraints

No negative consumption:

$$c_t \geq 0. \tag{4}$$

Budget constraint: for $t = 0, 1, \ldots,$

$$c_t(D_t) + a_t(D_t) + \theta_t(D_t)s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) \leq w_t, \tag{5}$$

where, w_0 is given and, for $t = 1, 2, \ldots,$

$$\begin{aligned} w_t &= \theta_{t-1}(D_{t-1})(s_t(D_t) + (1-q)D_t) + \psi^1_{t-1}(D_{t-1}) \\ &\quad + \psi^2_{t-1}(D_{t-1})p^1_t(D_t) + \ldots + \psi^m_{t-1}(D_{t-1})p^{m-1}_t(D_t). \end{aligned}$$

The consumer is not allow to short or issue bonds so that $\psi_t(i) \in [0, B_t(i)]$ for all t and i.

w_t Constraints: matrix form

$$w_t = \theta_{t-1}(D_{t-1})(s_t(D_t) + (1-q)D_t) + \psi_{t-1}(D_{t-1}) \cdot (Ap_t(D_t) + b),$$

where A and b are defined as follows

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Model:Equilibrium quantities

In equilibrium: The consumer holds all the shares of stock and bonds at all time and consumes all the after-tax dividend of the stock and interest of the bonds in each period. That is, for all t,

$$\theta_t = 1, \psi_t = B_t$$

and

$$c_t(D_t) + a_t(D_t) = (1-q)D_t + B_{t-1} \cdot (Ap_t(D_t) + b) - B_t \cdot p_t(D_t).$$

Price process p_t (which determines the term structure) induces such a general market equilibrium is the equilibrium we are seeking.

Relationship between government and consumer

The quantity

$$g_t = B_t \cdot p_t(D_t) - B_{t-1} \cdot (Ap_t(D_t) + b)$$

is the government's additional expenditure financed by issuing bonds. When $g_t > 0$, the government consumes additional g_t from the consumer's portion of the dividend and when $g_t < 0$ the government yield part of its portion of the dividend to consumers.

Model: Optimization problem

$$
\begin{aligned}
\text{maximize} \quad & E\left[\sum_{t=0}^{\infty} \beta^t [u(c_t) + \phi(a_t - \gamma_t)]\right] \qquad (6) \\
\text{subject to} \quad & c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, t = 0, 1 \ldots. \\
& c_t(D_t) + a_t(D_t) + \theta_t(D_t) s_t(D_t) + \psi_t(D_t) \cdot p_t(D_t) \\
& \quad \leq \theta_{t-1}(D_{t-1})(s_t(D_t) + (1-q)D_t) \\
& \quad + \psi_{t-1}(D_{t-1}) \cdot (Ap_t(D_t) + b), t > 0, \\
& c_0(D_0) + a_0(D_0) + \theta_0(D_0) s_0(D_0) + \psi_0(D_0) \cdot p_0(D_0) \leq w_0.
\end{aligned}
$$

We assume that the initial state D_0 and the initial endowment w_0 are fixed.

Dynamic programming equations

Denote the optimal value function by $v(w_0, D_0, \vec{\gamma})$ where $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots)$. Using S to signify the right shifting operator we have dynamic programming equations, for $t = 0, 1, \ldots,$

$$\begin{aligned} & v(w, D, S^t\vec{\gamma}) = \qquad (7) \\ \max \quad & \left[u(c_t) + \phi(a_t - \gamma_t) + \beta \int v(w', D', S^{t+1}\vec{\gamma}) dF(D', D) \right] \\ \text{s.t.} \quad & c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, \\ & c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w \\ & \theta_t(s_{t+1}(D') + (1-q)D') + \psi_t \cdot (Ap_{t+1}(D') + b) = w'. \end{aligned}$$

Optimal value as functions of the portfolio

Given $\vec{s} = (s_0, s_1, \dots)$ and $\vec{p} = (p_0, p_1, \dots)$,

$$V(\theta, \psi, D, \vec{\gamma} | \vec{s}, \vec{p}) = v(\theta(s_0(D) + (1-q)D) + \psi \cdot (Ap_0(D) + b), D, \vec{\gamma}),$$

defines the optimal value as a function of the portfolio. We emphasis that

1. the portfolio (θ, ψ) is independent on D_{t+1}, since it is generated by the optimization problem for the period t and depends only on $D = D_t$, and
2. the price processes $\vec{s}$ and $\vec{p}$ relates the portfolio and the wealth level and, therefore, they must be specified in the definition of V.

Dynamic programming equations for portfolios

for $t = 0, 1, \ldots,$

$$\begin{aligned}
& V(\theta, \psi, D, S^t\vec{\gamma}|S^t\vec{s}, S^t\vec{p}) = \\
\max \quad & \left[u(c) + \phi(a - \gamma_t) + \beta \int V(\theta', \psi', D', S^{t+1}\vec{\gamma}|S^t\vec{s}, S^t\vec{p})dF(D', D)\right. \\
\text{s.t.} \quad & c \geq 0, \psi'(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, \\
& c + a + \theta' s_t(D) + \psi' \cdot p_t(D) \\
& \quad \leq \theta(s_t(D) + (1 - q)D) + \psi \cdot (Ap_t(D) + b).
\end{aligned} \tag{8}$$

Finite number of different consumption commitments

Under reasonable conditions we can solve for the equilibrium for the practically useful case in which there are only finite number of different consumption commitments, say all within n, using the following steps:

Finite equations

There are only $n+1$ different dynamical programming equations now. They are, for $t = 0, 1, \ldots, n-1$,

$$
\begin{aligned}
& v(w, D, S^t\vec{\gamma}) = && (9) \\
\max \quad & \left[u(c_t) + \phi(a_t - \gamma_t) + \beta \int v(w', D', S^{t+1}\vec{\gamma}) dF(D', D)\right] \\
\text{subject to} \quad & c_t \geq 0, \psi_t(i) \in [0, B_t(i)], i = 1, 2, \ldots, m, \\
& c_t + a_t + \theta_t s_t(D) + \psi_t \cdot p_t(D) \leq w \\
& \theta_t(s_{t+1}(D') + (1-q)D') + \psi_t \cdot (Ap_{t+1}(D') + b) = w'.
\end{aligned}
$$

The last equation

$$
\begin{aligned}
& v(w, D, \gamma\vec{1}) = \qquad\qquad (10) \\
\max \quad & \left[u(c_n) + \phi(a_n - \gamma_n) + \beta \int v(w', D', \gamma\vec{1}) dF(D', D)\right] \\
\text{subject to} \quad & c_n \geq 0, \psi_n(i) \in [0, B_n(i)], i = 1, 2, \ldots, m, \\
& c_n + a_n + \theta_n s_n(D) + \psi_n \cdot p_n(D) \leq w \\
& \theta_n(s_n(D') + (1-q)D') + \psi_n \cdot (Ap_n(D') + b) = w'.
\end{aligned}
$$

The uniqueness of the value function

The last equation determines $v(w, D, \gamma\vec{1})$ as a fixed point of a contraction.Then we can determine $v(w, D, S^t\vec{\gamma})$ consecutively using the first n equations.

Properties of the value function

Assuming u and ϕ are smooth and the maximum in dynamic programming are attained, we have $w \to v(w, D, \vec{\gamma})$ is differentiable and

$$v_w(w, D, \vec{\gamma}) = u'(c) = \phi'(a - \gamma_0). \tag{11}$$

Moreover $w \to v(w, D, \vec{\gamma})$ is an increasing concave function.

Definition of the equilibrium

Equilibrium

Let $\vec{\gamma}$ be a consumption commitment with $\gamma_t = 0$ for $t \geq n$. An equilibrium is a pair of stock and bond prices processes that $\vec{s}, \vec{p}$ and a continuous bounded optimal value function $V(\theta, \psi, D, \vec{\gamma}|\vec{s}, \vec{p})$ satisfying

(i) $S^t\vec{p} = S^{t'}\vec{p}, S^t\vec{s} = S^{t'}\vec{s}$, for $t, t' \geq n$,

(ii) dynamic programming system of equations, and

(iii) for each D and $t = 0, 1, \ldots, n$, $V(1, B_t, D, S^t\vec{\gamma}|S^t\vec{s}, S^t\vec{p})$ is attained by $\theta' = 1, \psi' = B_t$, and $c + a = (1-q)D + B_{t-1} \cdot (Ap_t(D) + b) - B_t \cdot p_t(D)$.

Characterizing the equilibrium

Using the properties of the optimal value function the optimality condition for the dynamic programming equations characterizes the equilibrium.

Equation systems characterizing the equilibrium

For $t = 0, 1, \ldots, n-1$,

$$\begin{aligned} & u'(c_t(D, p_t))(s_t(D), p_t(D)) \\ = \ & \beta \int u'(c_{t+1}(D', p_{t+1}))(s_{t+1}(D') + (1-q)D', \\ & Ap_{t+1}(D') + b)dF(D', D) \end{aligned} \tag{12}$$

and

$$\begin{aligned} & u'(c_n(D, p_n))(s_n(D), p_n(D)) \\ = \ & \beta \int u'(c_n(D', p_n))(s_n(D') + (1-q)D', Ap_n(D') + b)dF(D', D). \end{aligned} \tag{13}$$

Existence and uniqueness

Existence is easier: impose condition so that the Schauder's fixed point theorem applies to the equation system characterizing the equilibrium.

Uniqueness is harder: impose condition so that the Banach fixed point theorem applies to the equation system characterizing the equilibrium.

Modeling right those conditions are satisfied.

Computing the equilibrium

Banach's iterative algorithm for contractions.

Parameters, utility and penalty functions

We use parameters that mimic the US bond markets: assuming 1 to 30 year bonds with heavy short term (1 year) liquidity.
We use the following utility and penalty functions

$$u(t) = \ln(t)$$

and

$$\phi(t) = \frac{2}{k^2} t^2.$$

Without consumption commitment

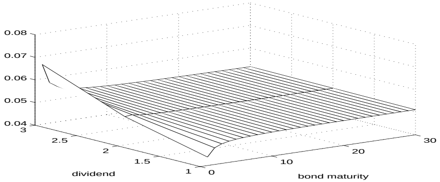

Figure: Yield surface: no consumption commitment

Typical consumption commitments

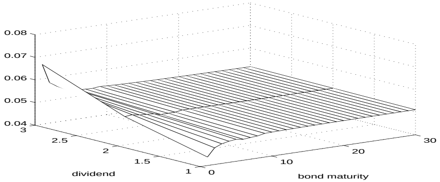

Figure: Yield surface: consumption commitment at $t = 6$

Impact of one consumption commitment at 8: no addition supply of bonds

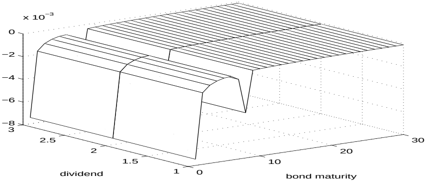

Figure: Deviation due to a single consumption commitment

Observations

- the yield of 8 year bond is much lower then other bonds due to the consumption commitment;
- the yields of the other bonds are also depressed,
- the yield reductions for bonds with maturity close to the 8 year bonds are reduced more;
- the yield of the bonds with very short maturity – 1,2,3 are heavily depressed –due to the formula

$$\frac{1}{i} \ln p_0(i)$$

  and
- the depression of the yield for bonds with maturity shorter than 8 are heavier than those with maturity longer than 8.

The role of liquidity

The effect we see in the previous picture is the combination of the heavy short term (1 year) liquidity and the liquidity reduction at the time of the commitment. Taking away these we have

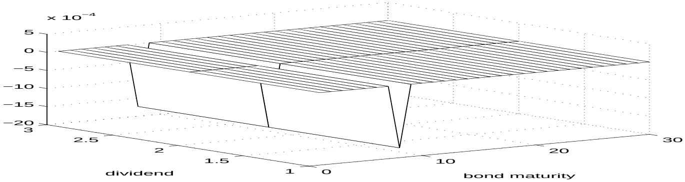

Figure: Isolating the effect of consumption commitment

Conclusion

1. Consumption commitments at t reduce the yields of the t year bonds and those close to it.
2. The term structure is the result of multiple factors: consumption commitments, the status of the economy, and liquidity.
3. Higher comparative liquidity for t year bonds causes a substitution effect of period t. Substitution effect we observe in real markets is due to the high liquidity of short term debt instruments.
4. Providing additional liquidity for a bond in unusually high demand by the government is an effective way of minimizing its impact.

Problems for further study

1. Nonsmooth utility and penalty functions.
2. Role of duality.
3. Relationship with other models.
4. Model real markets.