Asymptotics of implied volatility in local volatility models

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Joint work with

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References

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Outline

- Implied volatility in terms of local volatility
  - The heat kernel approach
  - The BBF approximation
  - BBF to higher orders
- One expansion, two approaches
  - Laplace asymptotic formula
  - Expansion of time value
- Numerical tests
- Summary and conclusions

Objective

Given a local volatility process

$$\frac{dS}{S} = \sigma(S,t)\,dW_t,$$

with $\sigma(S,t)$ depending only on the underlying level S and the time t, we want to compute implied volatilities $\sigma_{bs}(K,T)$ such that

$$C_{bs}(s,t,K,T,\sigma_{bs}(K,T)) = \mathbb{E}\left[(S_T - K)^+ | S_t = s\right]$$

or in words, we want to efficiently compute implied volatility from local volatility.

Call price

Let $p(t, s; t', s')$ be the transition probability density. Then

$$\begin{aligned} C(s,t,K,T) &= \mathbb{E}\left[(S_T - K)^+ | S_t = s\right] \\ &= \int (s' - K)^+ p(t, s; T, s') ds' \end{aligned}$$

As a function of t and s, p satisfies the backward Kolmogorov equation:

$$Lp := p_t + \frac{1}{2} s^2 \sigma^2(s,t) p_{ss} = 0,$$

Subindices refer to respective partial derivatives.

Two to approximate

$$C(s,t,K,T) \quad = \quad \int (s'-K)^{+} p(t,s;T,s')ds'$$

- Approximate transition density by heat kernel expansion.
- Approximate the integral.
    - Two approaches for approximating the integral lead to one expansion.
- The smaller the time to maturity, the better the approximation, for both approximations.

Heat kernel expansion

Heat kernel expansion for transition density $p(t,s;t',s')$ when $t'-t$ is small:

$$p(t,s;t',s') \sim \frac{e^{-\frac{d^2(s,s',t)}{2(t'-t)}}}{\sqrt{2\pi(t'-t)}s'\sigma(s',t')}\left[\sum_{k=0}^{n} H_k(t,s,s')(t'-t)^k\right]$$

- $d(s,s',t) = \left|\int_s^{s'} \frac{d\xi}{\xi\sigma(\xi,t)}\right|$: geodesic distance between s to s'
- $H_0(t,s,s') = \sqrt{\frac{s\sigma(s,t)}{s'\sigma(s',t)}} \exp\left[\int_s^{s'} \frac{d_t(\eta,s',t)}{\eta\sigma(\eta,t)} d\eta\right]$
- $H_i(t,s,s') = \frac{H_0(t,s,s')}{d^i(s,s',t)} \int_{s'}^{s} \frac{d^{i-1}(\eta,s',t)LH_{i-1}}{H_0(\eta,s',t)a(\eta,t)} d\eta$

Heat kernel expansion for Black-Scholes

Heat kernel expansion for Black-Scholes transition density $p_{bs}(t, s; t', s')$ when $t' - t$ is small:

$$p_{bs}(t'-t, s, s') = \frac{e^{-\frac{d_{bs}^2(s,s')}{2(t'-t)}}}{\sqrt{2\pi(t'-t)}\sigma_{bs}s'}\sqrt{\frac{s}{s'}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\frac{\sigma_{bs}^2(t'-t)}{8}\right]^k$$

- $d_{bs}(s, s') = \left|\int_s^{s'} \frac{d\xi}{\sigma_{bs}\xi}\right| = \frac{1}{\sigma_{bs}}\left|\log\frac{s'}{s}\right|$
- $H_0^{bs}(t, s, s') = \sqrt{\frac{s}{s'}}$

Main idea

Implied volatility σ_{bs} is defined as the unique solution to

$$C(s,t,K,T) = C_{bs}(s,t,K,T,\sigma_{bs})$$

- Substitute the transition density by the heat kernel expansion for both the model price C and the Black-Scholes price C_{bs}
- Expand in terms of $T-t$ on both sides of the resulting equation
- Further expand on Black-Scholes side the implied volatility

$$\sigma_{bs}(K,T) \approx \sigma_{bs,0} + \sigma_{bs,1}(T-t) + \sigma_{bs,2}(T-t)^2$$

- Match the corresponding coefficients

Two approaches

- Directly substitute the transition density by heat kernel expansion to call price. Use Laplace asymptotic formula to approximate the resulting integral.
- Rewrite call price as intrinsic value + time value. Further rewrite time value as an integral of transition density over time, i.e., the Carr-Jarrow formula:

$$C(s,t,K,T) = (s-K)^{+} + \int_t^T K^2\sigma^2(K,u)p(s,t;K,u)du$$

Laplace asymptotic formula

Asymptotic expansion of the integral as $\tau \to 0^+$

$$\int_0^\infty e^{-\frac{\phi(x)}{\tau}} f(x)dx \sim \tau^2 e^{-\frac{\phi(x^*)}{\tau}} \left[\frac{f'(x^*)}{[\phi'(x^*)]^2} + \left(\frac{f'(x^*)}{[\phi'(x^*)]^3} \right)' \tau \right]$$

Assumptions:

- f is identically zero when $0 \leq x \leq x^*$.
- ϕ is increasing in $[x^*, \infty)$.

Laplace asymptotic for call price

Let $\tau = T - t$.

$$C(s,t,K,T) = \int_0^{\infty} (s-K)^+ p(t,s;T,s')ds'$$

$$\sim \frac{1}{\sqrt{2\pi\tau}} \int_0^{\infty} (s'-K)^+ \frac{e^{-\frac{d^2(s,s',t)}{2\tau}}}{s'\sigma(s',T)} \sum_{k=0}^{n} H_k(t,s,s')\tau^k ds'$$

$$= \frac{1}{\sqrt{2\pi\tau}} \sum_{k=0}^{n} \int_K^{\infty} e^{-\frac{d^2(s,s',t)}{2\tau}} G_k(t,s,T,s')ds' \cdot \tau^k$$

- $G_k(t,s,T,s') = (s'-K)\frac{H_k(t,s,s')}{s'\sigma(s',T)}$

Laplace asymptotic for call price

Assume $s < K$.

$$\frac{1}{\sqrt{2\pi\tau}} \int_K^\infty e^{-\frac{d^2(s,s',t)}{2\tau}} G_k(t,s,T,s')ds'$$
$$\sim \frac{\tau^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{d^2}{2T}} \left[\frac{G_k'}{(dd')^2} + \left(\frac{G_k'}{(dd')^3} \right)' \tau \right],$$

- $d = d(s,K,t)$, $d' = \frac{\partial d}{\partial s'}(s,K,t)$, and $d'' = \frac{\partial^2 d}{\partial (s')^2}(s,K,t)$
- $G_k' = \frac{\partial G_k}{\partial s'}(t,s,T,K) = \frac{H_k(t,s,K)}{K\sigma(K,T)}$

Laplace asymptotic for call price

Laplace asymptotic for model price:

$$C(s,t,K,T) \sim \frac{\tau^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{d^2}{2\tau}} \left[\frac{G_0'}{(dd')^2} + \left\{ \left(\frac{G_0'(K)}{(dd')^3} \right)' + \frac{G_1'(K)}{(dd')^2} \right\} \tau \right].$$

- $d = d(s,K,t)$, $d' = \frac{\partial d}{\partial s'}(s,K,t)$, and $d'' = \frac{\partial^2 d}{\partial (s')^2}(s,K,t)$
- $G_k' = \frac{\partial G_k}{\partial s'}(t,s,T,K) = \frac{H_k(t,s,K)}{K\sigma(K,T)}$

Laplace asymptotic for Black-Scholes: $k = \log \frac{K}{s}$

$$C_{bs}(s,t,K,T,\sigma_{bs}) \sim \frac{Ke^{-\frac{k}{2}}}{\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma_{bs}^2\tau}} \frac{\sigma_{bs}^3 \tau^{\frac{3}{2}}}{k^2} \left[1 - \left(\frac{1}{8} + \frac{3}{k^2} \right) \sigma_{bs}^2 \tau \right]$$

Match the coefficients

Let $\sigma_{bs} = \sigma_{bs,0} + \sigma_{bs,1}\tau + \sigma_{bs,2}\tau^2 + \cdots$ and set

$$\begin{aligned} & e^{-\frac{d^2}{2\tau}}\left[\frac{G_0'}{(dd')^2} + \left\{\left(\frac{G_0'(K)}{(dd')^3}\right)' + \frac{G_1'(K)}{(dd')^2}\right\}\tau\right] \\ = \quad & e^{-\frac{k^2}{2\sigma_{bs}^2\tau}}\frac{K\sigma_{bs}^3}{k^2 e^{\frac{k}{2}}}\left[1 - \left(\frac{1}{8} + \frac{3}{k^2}\right)\sigma_{bs}^2\tau\right] \end{aligned}$$

- Exponential term: $d^2 = \frac{k^2}{\sigma_{bs,0}^2} \implies \sigma_{bs,0} = \frac{k}{d} = \frac{\log K - \log s}{d(s,K,t)}$
- Zeroth order term:

$$\frac{G_0'}{(dd')^2} = e^{\frac{k^2\sigma_{bs,1}}{\sigma_{bs,0}^3}}\frac{K\sigma_{bs,0}^3}{k^2 e^{\frac{k}{2}}} \Longrightarrow \sigma_{bs,1} = \frac{k}{d^3}\log\left[\frac{dG_0' e^{-\frac{k}{2}}}{Kk(d')^2}\right]$$

Time value

Recall

$$
\begin{aligned}
C(s,t,K,T) &= (s-K)^+ + \int_t^T K^2\sigma^2(K,u)p(s,t;K,u)du \\
\sim\ & (s-K)^+ + \sum_{k=0}^{n}\int_t^T \frac{e^{-\frac{d^2(s,K,t)}{2(u-t)}}}{\sqrt{2\pi(u-t)}} K\sigma(K,u)(u-t)^k du \cdot H_k(t,s,K)
\end{aligned}
$$

Moreover, denote $d = d(s,K,t)$,

$$
\begin{aligned}
& \int_t^T e^{-\frac{d^2}{2(u-t)}}\sigma(K,u)(u-t)^{k-\frac{1}{2}}du \\
\sim\ & \int_t^T e^{-\frac{d^2}{2(u-t)}}[\sigma(K,t)+\sigma_t(K,t)(u-t)](u-t)^{k-\frac{1}{2}}du
\end{aligned}
$$

Expansion for call price

Let $\Phi_k(d,\tau) = \int_0^t u^{k-\frac{1}{2}} e^{-\frac{d^2}{2u}} du$.

$$
\begin{aligned}
& C(s,t,K,T) - (s-K)^+ \\
\sim \quad & \frac{1}{2\sqrt{2\pi}} \{K\sigma(K,t)\Phi_0(d,\tau)H_0(t,s,K) \\
& +K[\sigma_t(K,t)H_0(t,s,K) + \sigma(K,t)H_1(t,s,K)]\Phi_1(d,\tau)\}
\end{aligned}
$$

Moreover, on Black-Scholes side,

$$
\begin{aligned}
& C_{bs}(s,t,K,T) - (s-K)^+ \\
\sim \quad & \frac{\sqrt{sK}}{2\sqrt{2\pi}} \left[\sigma_{bs}\Phi_0(d_{bs},\tau) - \frac{\sigma_{bs}^3}{8}\Phi_1(d_{bs},\tau)\right]
\end{aligned}
$$

Auxiliary expansion and matching

Expanding the Φ_i's:

- $\Phi_0(d,\tau) \sim 2\tau^{\frac{3}{2}}\left[\frac{1}{d^2} - 3\frac{\tau}{d^4}\right]e^{-\frac{d^2}{2\tau}}$
- $\Phi_1(d,\tau) = \frac{2}{3}\tau^{\frac{3}{2}}e^{-\frac{d^2}{2\tau}} - \frac{d^2}{3}\Phi_0(d,\tau) \sim \frac{2\tau^{\frac{5}{2}}}{d^2}e^{-\frac{d^2}{2\tau}}$

Matching

$$
\begin{aligned}
& e^{-\frac{d^2(s,K,t)}{2\tau}}\left\{\frac{K\sigma H_0}{d^2} + \left[\frac{K\sigma_t H_0 + K\sigma H_1}{d^2} - \frac{3K\sigma H_0}{d^4}\right]\tau\right\} \\
= \quad & e^{-\frac{d_{bs}^2(s,K,t)}{2\tau}}\sqrt{sK}\left[\sigma_{bs}\Phi_0(d_{bs},\tau) - \frac{\sigma_{bs}^3}{8}\Phi_1(d_{bs},\tau)\right]
\end{aligned}
$$

Asymptotic expansion once again

$$\sigma_{bs} = \sigma_{bs,0} + \sigma_{bs,1}(T-t) + \sigma_{bs,2}(T-t)^2 + \mathcal{O}(T-t)^3.$$

$$d(s,K,t) = \int_s^K \frac{d\xi}{\xi\sigma(\xi,t)},$$

$$H_0(s,K,t) = \sqrt{\frac{s\sigma(s,t)}{K\sigma(K,t)}}\exp\left[\int_s^K \frac{d_t(\eta,K,t)}{\eta\sigma(\eta,t)}d\eta\right].$$

- $\sigma_{bs,0} = \dfrac{|\log K - \log s|}{d(s,K,t)}$. (BBF)
- $\sigma_{bs,1} = \dfrac{k}{d^3}\log\left[\dfrac{dH_0\sqrt{K}\sigma(K,t)}{k\sqrt{s}}\right]$, where $k = \log K - \log s$.
- $\sigma_{bs,2}$? Too complicated to reproduce here.

Henry-Labordère's approximation

Henry-Labordère also presents a heat kernel expansion based approximation to implied volatility in equation (5.40) on page 140 of his book [4]:

$$\sigma_{BS}(K,T) \approx \sigma_0(K)\left\{1 + \frac{T}{3}\left[\frac{1}{8}\sigma_0(K)^2 + \mathcal{Q}(f_{av}) + \frac{3}{4}\mathcal{G}(f_{av})\right]\right\} \tag{1}$$

with

$$\mathcal{Q}(f) = \frac{C(f)^2}{4}\left[\frac{C''(f)}{C(f)} - \frac{1}{2}\left(\frac{C'(f)}{C(f)}\right)^2\right]$$

and

$$\mathcal{G}(f) = 2\,\partial_t \log C(f) = 2\,\frac{\partial_t\,\sigma(f,t)}{\sigma(f,t)}$$

where $C(f) = f\,\sigma(f,t)$ in our notation, $f_{av} = (S_0 + K)/2$ and the term $\sigma_0(K)$ is the BBF approximation from [1].

How well do these approximations work?

We consider the following explicit local volatility models:

- The square-root CEV model:

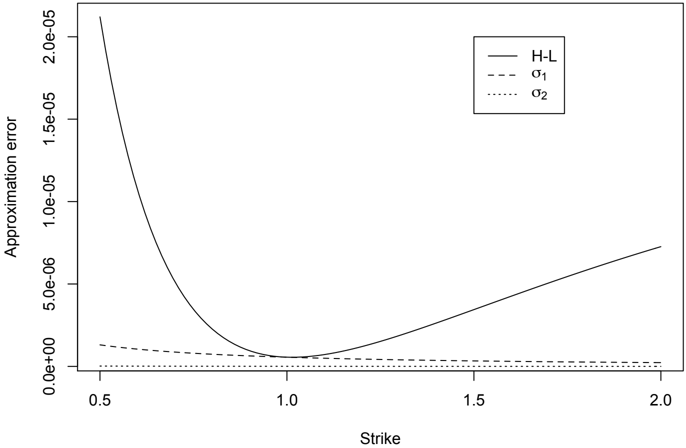
$$dS_t = e^{-\lambda\, t}\, \sigma\, \sqrt{S_t}\, dW_t$$

- The quadratic model:

$$dS_t = e^{-\lambda\, t}\, \sigma\, \left\{1 + \psi\,(S_t - 1) + \frac{\gamma}{2}\,(S_t - 1)^2\right\}\, dW_t$$

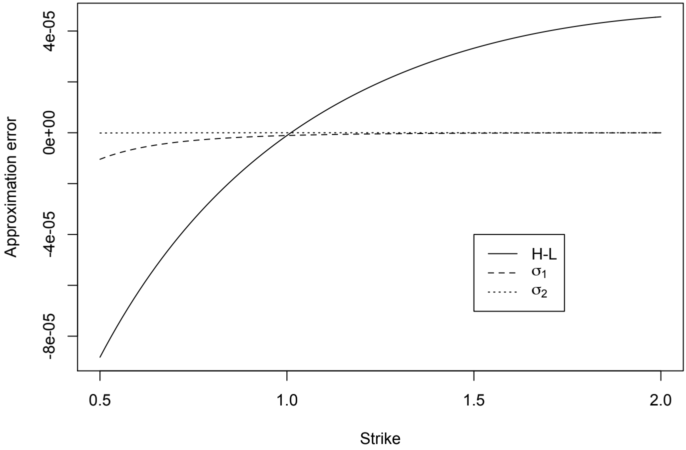
- Parameters are: $\sigma = 0.2$, $\psi = -0.5$ and $\gamma = 0.1$. In each case $S_0 = 1$ and $T = 1$.
- $\lambda = 0$ gives a time-homogeneous local volatility surface and $\lambda = 1$ a time-inhomogeneous one.
- We compare implied volatilities from the approximations and the closed-form solution.

Time-homogeneous Square Root CEV

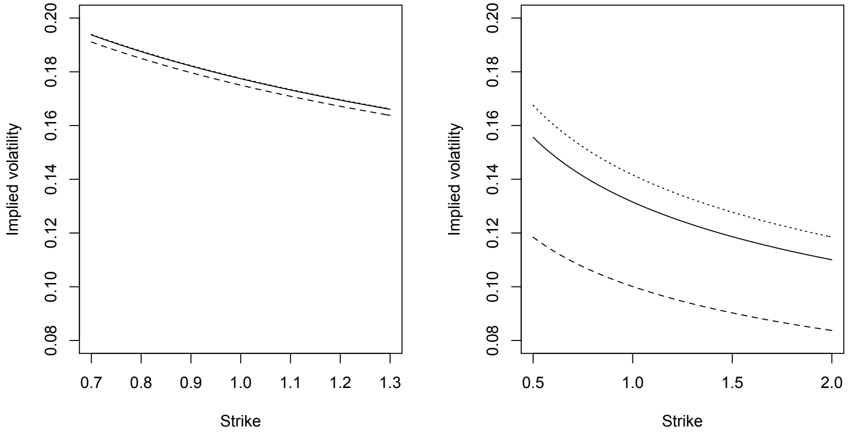


Note that all errors are tiny!

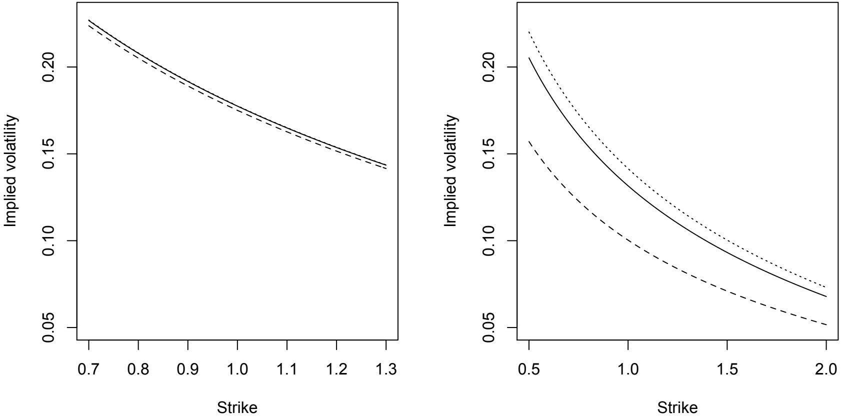
Time-homogeneous Quadratic Model



Time-inhomogeneous Square Root CEV



Time-inhomogeneous Quadratic Model



Summary

- Small-time expansions are useful for generating closed-form expressions for implied volatility from simple models.
- Direct substitute approach is easier for generalization to higher dimensions, e.g., stochastic volatility models.

$$\sigma_{bs} \sim \frac{\log K - \log s}{d_M(s, v)},$$

where $d_M(s, v)$ is the "distance to the money", i.e., shortest geodesic distance from the spot (s, v) to the line $\{s = K\}$ in the price-volatility plane.

  - Application: Short time implied vol in delta is flat! (Joint work with Carr and Lee).

Summary II

- Time value approach is easier for getting higher order terms.
- Refinement of $\sigma_{bs,0}$ (work in progress with Gatheral):

$$\sigma_{bs} \sim \left[\frac{\sqrt{T-t}}{|\log K - \log s|} \sqrt{\int_t^T \left| \frac{s'(\tau)}{a(s(\tau),\tau)} \right|^2 d\tau} \right]^{-1}$$

where the integral is along the "most likely path" $s(\tau)$.

  - If we take the "likely path" as $s(\tau) = \varphi_t^{-1}\left(\frac{\tau}{T}\varphi_t(K)\right)$, where $\varphi_t(x) = \int_s^x \frac{d\xi}{a(\xi,t)}$, then BBF is recovered.

THANK YOU FOR YOUR PATIENCE.