Forward is backward for time-homogeneous diffusions
or
Why worry about boundary conditions?

Johan Tysk
Joint work with Erik Ekström
Department of Mathematics, Uppsala University

June 26, 2010
6th World Congress of the Bachelier Finance Society
Toronto

In physics, spatial boundary conditions for the heat equation on a half-line (half-space) are always provided.

In finance such boundary conditions are often omitted for pricing equations, perhaps since for geometric Brownian motion such conditions are redundant.

For many models in finance, however, boundary conditions are needed and even if they are redundant, boundary conditions (boundary behaviour) provides useful information.

We will here not focus on when the conditions are redundant or not, rather on the information they provide. This is particularly striking in the case when we consider forward equations whose boundary behaviour can be reduced, in the case of time-homogeneous diffusions, to the study of boundary behaviour for backward equations.

Feynman-Kac

We study diffusions on the half-line satisfying

$$dX_t = \beta(X_t)\,dt + \sigma(X_t)\,dW.$$

Consider an expected value of the form

$$u(x,t) = E_{x,t}g(X_T).$$

The corresponding partial differential equation is

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} + \beta(x)u_x = 0 \\ u(x,T) = g(x). \end{cases}$$

Feynman-Kac

We study diffusions on the half-line satisfying

$$dX_t = \beta(X_t)\,dt + \sigma(X_t)\,dW.$$

Consider an expected value of the form

$$u(x,t) = E_{x,t}\,g(X_T).$$

The corresponding partial differential equation is

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} + \beta(x)u_x = 0 \\ u(x,T) = g(x). \end{cases}$$

What about boundary conditions at $x = 0$?

An example: The Black-Scholes model

Stock price:

$$dX_t = rX_t\,dt + \sigma X_t\,dW$$

Call price:

$$u(x,t) = E_{x,t}e^{-r(T-t)}(X_T - K)^+$$

The BS-equation:

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \\ u(x,T) = (x-K)^+. \end{cases}$$

An example: The Black-Scholes model

Stock price:

$$dX_t = rX_t\,dt + \sigma X_t\,dW$$

Call price:

$$u(x,t) = E_{x,t}e^{-r(T-t)}(X_T - K)^+$$

The BS-equation:

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \\ u(x,T) = (x-K)^+. \end{cases}$$

Boundary condition at $x = 0$:

$$u(0,t) = 0$$

Another example: The CIR-process

Assume that

$$dX_t = (b - aX_t)\,dt + \sigma\sqrt{X_t}\,dW,$$

and

$$u(x,t) = E_{x,t}\,g(X_T).$$

The corresponding partial differential equation is

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} + \beta(x)u_x = 0 \\ u(x,T) = g(x). \end{cases}$$

What about boundary conditions at $x = 0$?

The CIR-process, continued

- If $b < \sigma^2/2$, then 0 is reached with positive probability. Boundary conditions are needed to ensure uniqueness of solutions. Indeed, Heston-Loewenstein-Willard (2007) provide two different continuous bounded solutions to the term structure equation (with no specified boundary conditions).
- If $b > \sigma^2/2$, then 0 cannot be reached. Boundary conditions are not needed from a mathematical perspective, but crucial from a numerical perspective!

Boundary conditions for the backward equation

Hypothesis

Assume that $\sigma^2(x,t)$ *and* $\beta(x,t)$ *are* C^1 *in* x*,* $\sigma(x,t)=0$ *iff* $x=0$*,* $\beta(0,t)\geq 0$ *and linear growth conditions. Let* g *be* C^1 *with* g *and* g' *bounded.*

Theorem

The function $u(x,t)=E_{x,t}g(X(T))$ *is the unique classical solution to the backward equation*

$$\begin{cases} u_t+\frac{1}{2}\sigma^2 u_{xx}+\beta u_x=0 \\ u(x,T)=g(x) \\ u_t(0,t)+\beta(0,t)u_x(0,t)=0. \end{cases}$$

Note: A classical solution is understood to be C^1 up to the boundary $x = 0$, so the boundary condition $u_t(0, t) = \beta(0, t)u_x(0, t)$ holds in a classical sense.

Stochastic volatility models

Stock price:

$$\begin{cases} dX(t) = \sqrt{Y(t)}\alpha(X(t))\,dW \\ dY(t) = \beta(Y(t))\,dt + \sigma(Y(t))\,dV, \end{cases}$$

$dW\,dV = \rho\,dt$.
We assume that X is absorbed at 0, and that $\beta(0) \geq 0$, $\sigma(0) = 0$ and linear growth conditions.

Theorem

The function $u(x,y,t) = E_{x,y,t}g(X(T))$ is the unique classical solution to the backward equation

$$\begin{cases} u_t + \frac{1}{2}x^2 y u_{xx} + \rho x\sqrt{y}\sigma u_{xy} + \frac{1}{2}\sigma^2 u_{yy} + \beta u_y = 0 \\ u(x,y,T) = g(x) \\ u(0,y,t) = g(0) \\ u_t(x,0,t) + \beta(0)u_y(x,0,t) = 0. \end{cases}$$

The Kolmogorov (Fokker-Planck) forward equation

Time-homogeneous diffusion:

$$dX_t = \beta(X_t)\,dt + \sigma(X_t)\,dW$$

with $b(0) \geq 0$ and $\sigma(0) = 0$
Density:

$$p(x,y,t) := P_x(X_t \in dy)/dy$$

The density satisfies the <u>backward</u> equation

$$\begin{cases} p_t = \frac{1}{2}\sigma^2 p_{xx} + \beta p_x \\ p(x,y,0) = \delta_y(x) \\ p_t(0,y,t) = \beta(0) p_x(0,y,t). \end{cases}$$

The density also satisfies the <u>forward</u> equation

$$\begin{cases} p_t = \frac{1}{2}(\sigma^2 p)_{yy} - (\beta p)_y \\ p(x,y,0) = \delta_x(y). \end{cases}$$

The density also satisfies the forward equation

$$\begin{cases} p_t = \frac{1}{2}(\sigma^2 p)_{yy} - (\beta p)_y \\ p(x,y,0) = \delta_x(y). \end{cases}$$

What about boundary conditions?

The CIR-process

$$dX_t = (b - aX_t)\,dt + \sigma\sqrt{X_t}\,dW$$

- If $b > \sigma^2/2$, then $\lim_{y\to 0} p(x,y,t) = 0$. Thus $p(x,0,t) = 0$ can be used as boundary condition.

The CIR-process

$$dX_t = (b - aX_t)\,dt + \sigma\sqrt{X_t}\,dW$$

- If $b > \sigma^2/2$, then $\lim_{y\to 0} p(x,y,t) = 0$. Thus $p(x,0,t) = 0$ can be used as boundary condition.
- If $b < \sigma^2/2$, then $\lim_{y\to 0} p(x,y,t) = \infty$. It is then unclear what boundary conditions to impose.

A Symmetry Relation

Let $m(x) = \frac{2}{\sigma^2(x)} \exp\{\int^x \frac{2\beta(z)}{\sigma^2(z)}\, dz\}$ be the density of the speed measure.

Theorem

The density satisfies

$$m(x)p(x, y, t) = m(y)p(y, x, t).$$

A Symmetry Relation

Let $m(x) = \frac{2}{\sigma^2(x)} \exp\{\int^x \frac{2\beta(z)}{\sigma^2(z)} dz\}$ be the density of the speed measure.

Theorem

The density satisfies

$$m(x)p(x,y,t) = m(y)p(y,x,t).$$

Note: Using the symmetry relation, we circumvent the difficult issue of boundary conditions for the forward equation. Instead, we solve the backward equation (with boundary conditions specified above), and then use the symmetry relation.

Boundary asymptotics

As a by-product we also get the exact boundary asymptotics.

Theorem

The density $p(x,y,t)$ satisfies $p(x,y,t)/m(y) \to C(t)$ as $y \to 0$ for some C^1 function $C(t) := p(0,x,t)/m(x) \geq 0$.

(i) *If $\beta(0) > 0$, then $C(t)$ is positive for $t > 0$.*

(ii) *If $\beta(0) = 0$, then the function $C(t) \equiv 0$. In this case, define*

$$D(t) = \lim_{y \downarrow 0} \frac{p(x,y,t)}{ym(y)}.$$

(iia) *If there exists a constant $\varepsilon > 0$ such that $\sigma(x) \geq \varepsilon x^{1-\varepsilon}$ for $0 < x < \varepsilon$, then $D(t)$ is positive for $t > 0$.*

(iib) *If $\beta(0) = 0$ and there exists $\varepsilon > 0$ such that $\sigma(x) \leq \varepsilon^{-1} x$ for $x \in (0,\varepsilon)$, then $D \equiv 0$.*

Sketch of proof

The above symmetry relation was also proved by Ito-McKean. Their proof is rather sketchy and relies on spectral decomposition of solutions to PDEs. Our proof is more elementary and uses the known regularity in the backward variable and integration by parts.

Fix x and let $\tilde{p}(y,t) := \frac{m(y)}{m(x)} p(y,x,t)$, and let

$$u(y,t) = E_{y,t} g(X_T)$$

for some $T > 0$ and some function g.

Integration by parts:

$$\int_0^\infty \int_0^T u_t(y,t)\tilde{p}(y,t)\,dtdy =$$
$$\int_0^\infty u(y,T)\tilde{p}(y,T) - u(y,0)\tilde{p}(y,0)\,dy - \int_0^\infty \int_0^T u(y,t)\tilde{p}_t(y,t)\,dtdy.$$

and

$$\begin{aligned}
& -\int_0^\infty \int_0^T u_t(y,t)\tilde{p}(y,t)\,dtdy \\
= & \int_0^\infty \int_0^T \big(\alpha u_{yy}(y,t) + \beta u_y(y,t)\big)\,\tilde{p}(y,t)\,dtdy \\
= & \int_0^T \int_0^\infty u(y,t)\tilde{p}_t(y,t)\,dydt.
\end{aligned}$$

Thus

$$\int_0^\infty u(y,0)\tilde{p}(y,0)\,dy = \int_0^\infty u(y,T)\tilde{p}(y,T)\,dy.$$

Since $\tilde{p}(y,0) = \delta_x(y)$ and $u(y,T) = g(y)$, we have

$$E_{x,0}g(X_T) = \int_0^\infty g(y)\tilde{p}(y,T).$$

This holds for any function g, so $\tilde{p}(y,T)$ is a density of X_T.

Example: CIR

$$dX_t = (b - aX_t)\,dt + \sigma\sqrt{X_t}\,dW$$

$$m(x) = \frac{2}{\sigma^2} x^{2b/\sigma^2 - 1} \exp\left\{\frac{2a(1-x)}{\sigma^2}\right\}$$

The density satisfies

$$p(x,y,t) = \frac{m(y)}{m(x)} p(y,x,t) \sim C(x,t) y^{2b/\sigma^2 - 1}$$

for small y.

Example: CEV

The Constant Elasticity of Variance model is given by

$$dX_t = \sigma X_t^{\gamma} dW$$

for $\frac{1}{2} \leq \gamma < 1$. Now $m(x) = \frac{2}{\sigma^2} x^{-2\gamma}$. The density satisfies

$$p(x,y,t) = \frac{m(y)}{m(x)} p(y,x,t) \sim C(x,t) y^{1-2\gamma}$$

for small y.

References

- Ekström, Lötstedt, Tysk. Boundary values and finite difference methods for the single factor term strucure equation. *Appl. Math. Finance* (2009).
- Ekström and Tysk. Boundary conditions for the single factor term structure equation. *Ann. Appl. Probab*. (2010).
- Ekström and Tysk. The Black-Scholes equation in stochastic volatility models. *J. Math. Anal. Appl*. (2010).
- Ekström and Tysk. Forward is backward for time-homogeneous diffusions. Manuscript (2010).
- Heston, Loewenstein, Willard. Options and bubbles. *Rev. Financial Studies* (2007).

References

- Ekström, Lötstedt, Tysk. Boundary values and finite difference methods for the single factor term strucure equation. *Appl. Math. Finance* (2009).
- Ekström and Tysk. Boundary conditions for the single factor term structure equation. *Ann. Appl. Probab*. (2010).
- Ekström and Tysk. The Black-Scholes equation in stochastic volatility models. *J. Math. Anal. Appl*. (2010).
- Ekström and Tysk. Forward is backward for time-homogeneous diffusions. Manuscript (2010).
- Heston, Loewenstein, Willard. Options and bubbles. *Rev. Financial Studies* (2007).

Thank you for your attention!