Asymptotic Analysis for Stochastic Volatility:

Edgeworth expansion

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[Outline]

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[Model and basic idea]

We suppose that a log price process Z satisfies

$$\begin{aligned} \mathrm{d}Z_t &= \left\{ r_t - \frac{1}{2}\varphi(X_t)^2 \right\} \mathrm{d}t + \varphi(X_t) \left[\rho(X_t)\mathrm{d}W_t^1 + \sqrt{1-\rho(X_t)^2}\mathrm{d}W_t^2 \right] \\ \mathrm{d}X_t &= b(X_t)\mathrm{d}t + c(X_t)\mathrm{d}W_t^1, \end{aligned} \tag{1}$$

where

a) (W^1, W^2); 2-dimensional standard BM,
b) $r = \{r_t\}$; interest rate, deterministic,
c) b, c, ρ, φ; Borel functions.

The price $P[f]$ of the European option with payoff function $f \circ \log$ and maturity T is given by

$$P[f] = D\mathbb{E}[f(Z_T)], \quad D = \exp\left\{ -\int_0^T r_t \mathrm{d}t \right\}.$$

[Model and basic idea]

- Empirical studies show ρ appears “negative” and X appears “ergodic”.
- Explicit expression of the price is not available in general,
- while fast calibration and pricing are necessary in practice.
- An accurate approximation is useful and various asymptotic expansion methods have been proposed.
- Our purpose here is to present a new approximation formula which extends the so-called fast mean reverting singular perturbation formula, and to prove its validity.

[Model and basic idea]

Fix $\epsilon > 0$. By changing time scale $s = \epsilon^{-2}t$, we have that

$$
\begin{aligned}
Z_T &= Z_0 + \epsilon^2 \int_0^{T/\epsilon^2} \left\{ \hat{r}_s - \frac{1}{2}\varphi(\hat{X}_s)^2 \right\} \mathrm{d}s \\
&\quad + \epsilon \int_0^{T/\epsilon^2} \varphi(\hat{X}_s) \left[\rho(\hat{X}_s)\mathrm{d}\hat{W}_s^1 + \sqrt{1-\rho(\hat{X}_s)^2}\mathrm{d}\hat{W}_s^2 \right] \\
\mathrm{d}\hat{X}_s &= \epsilon^2 b(\hat{X}_s)\mathrm{d}s + \epsilon c(\hat{X}_s)\mathrm{d}\hat{W}_s^1,
\end{aligned}
$$

where $(\hat{W}_s^1, \hat{W}_s^2) = (\epsilon^{-1}W_{\epsilon^2 s}, \epsilon^{-1}W_{\epsilon^2 s})$, which is also an 2-dim standard BM.

If b and c are sufficiently “large”, then the law of $\hat{X}$ is “nondegenerate” even if $\epsilon > 0$ is small.

[Model and basic idea]

If in addition $\hat{X}$ is ergodic, or equivalently X is ergodic, then

$$\epsilon^2 \int_0^{T/\epsilon^2} \varphi(\hat{X}_t)^2 \mathrm{d}t \approx \Pi[\varphi^2]T, \quad \epsilon \int_0^{T/\epsilon^2} \varphi(\hat{X}_t) \mathrm{d}\check{W}_t \approx \mathcal{N}(0, \Pi[\varphi^2]T),$$

for small $\epsilon > 0$, by martingale CLT, where Π is the ergodic distribution of X.

As a result,

$$D\mathbb{E}[f(Z_T)] \approx D\mathbb{E}[f(Z_0 - \log(D) - \Sigma/2 + \sqrt{\Sigma}N)], \quad D = \exp\left\{-\int_0^T r_s \mathrm{d}s\right\},$$

where $N \sim \mathcal{N}(0,1)$, $\Sigma = \Pi[\varphi^2]T$. The right-hand expectation is the Black-Scholes price with volatility $\Pi[\varphi^2]^{1/2}$.

A correction term which improves this Black-Scholes approximation ?

[Model and basic idea]

Our main result is an error estimate for the approximation

$$D\mathbb{E}[f(Z_T)] \approx D\mathbb{E}[(1+p(N))f(Z_0 - \log(D) - \Sigma/2 + \sqrt{\Sigma}N)] \tag{2}$$

for every bounded Borel function f, where $N \sim \mathcal{N}(0,1)$, $\Sigma = \Pi[\varphi^2]T$ and

$$\begin{aligned} D &= \exp\left\{-\int_0^T r_s \mathrm{d}s\right\}, \\ p(z) &= \alpha\left\{1 - z^2 + \frac{1}{\sqrt{\Sigma}}(z^3 - 3z)\right\}, \\ \alpha &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left\{\frac{\varphi(v)^2}{\Pi[\varphi^2]} - 1\right\}\Pi(\mathrm{d}v)\frac{\varphi(x)\rho(x)}{c(x)}\mathrm{d}x. \end{aligned} \tag{3}$$

Here is an additional degree of freedom to capture the volatility skew.

[Singular perturbation expansion]

Fouque et al. (2000):

$$\begin{aligned}
\mathrm{d}Z_t &= \left\{r_t - \frac{1}{2}\varphi(X_t)^2\right\}\mathrm{d}t + \varphi(X_t)\left[\rho(X_t)\mathrm{d}W_t^1 + \sqrt{1-\rho(X_t)^2}\mathrm{d}W_t^2\right] \\
\mathrm{d}X_t &= \eta^{-2}b(X_t)\mathrm{d}t + \eta^{-1}c(X_t)\mathrm{d}W_t^1,
\end{aligned}$$

with $b(x) = a(b-x) + \eta\Lambda(x)$, $c(x) \equiv c$ and $\rho(x) \equiv \rho$.

Taylor expanding the price $P[f] = D\mathbb{E}[f(Z_T)]$ in η around 0, they obtained

$$P[f] = P_0[f] + \eta P_1[f] + \cdots$$

This is based on a singular perturbation of the PDE satisfied by $P[f]$.

[Singular perturbation expansion]

- The asymptotic expansion is around the Black-Scholes price, so an asymptotic expansion for the Black-Scholes implied volatility follows.
- Validated so far only when coefficients and payoff are sufficiently smooth.
- Error estimates are e.g., $O(\eta^2 \log \eta)$ for call/put options (Fouque et al. 2003), and $O(\eta^{4/3} \log \eta)$ for digital options (Fouque et al. 2005).
- Conlon and Sullivan (2005), Khasminskii and Yin (2005).
- We take a probabilistic approach, without using the artificial variable η.

[Edgeworth expansion (for IID)]

The Edgeworth expansion is a refinement of CLT, and is a rearrangement of the Gram-Charlier expansion.

Let $Z_n = n^{-1/2} \sum_{j=1}^{n} X_j$ with a standardized IID sequence X_j. If, e.g.,

- $\mathbb{E}[X_1^4] < \infty$, and
- $\limsup_{|u| \to \infty} |\mathbb{E}[\exp\{iuX_1\}]| = 0$,

then, uniformly in $z \in \mathbb{R}$,

$$\mathbb{P}[Z_n \le z] = \Phi(z) + \frac{E[X_1^3]}{6\sqrt{n}}(1 - z^2)\phi(z) + O(n^{-1}).$$

[Edgeworth expansion for ergodic diffusions]

Fukasawa (2008): we use the fact that

$$\{X_t\}_{\tau_j \le t \le \tau_{j+1}}, \quad \tau_{j+1} = \inf\left\{ t > \tau_j; X_t = x \text{ and } \sup_{\tau_j \le s \le t} X_s \ge y \right\}$$

$j = 1, 2, \dots$ are IID, where $x < y$ are fixed and $\tau_0 = 0$.

In particular,

$$\int_0^T f(X_t)\mathrm{d}t \approx \sum_{j=0}^{N_T} F_j, \quad F_j = \int_{\tau_j}^{\tau_{j+1}} f(X_t)\mathrm{d}t, \quad N_T = \{\max n; \tau_n \le T\}.$$

Since N_T is random so affects the first-order expansion, a technique for dealing with the joint distribution $(F_j, \tau_{j+1} - \tau_j)$ is required.

Other approaches: Yoshida (1997), Kusuoka and Yoshida (2000).

[Main theorem]

Define the scale function $s : \mathbb{R} \to \mathbb{R}$ and the normalized speed measure density $\pi : \mathbb{R} \to \mathbb{R}$ as

$$s(x) = \int_0^x \exp\left\{-2\int_0^v \frac{b(w)}{c(w)^2}\mathrm{d}w\right\}\mathrm{d}v, \quad \pi(x) = \frac{1}{\epsilon^2 s'(x)c^2(x)} \tag{4}$$

with

$$\epsilon^2 = \int \frac{\mathrm{d}x}{s'(x)c^2(x)}. \tag{5}$$

Note that $\mathcal{L}^X s = 0$. It is well-known that the stochastic differential equation for X in (1) has a unique weak solution which is ergodic if $\epsilon < \infty$ and $s(\mathbb{R}) = \mathbb{R}$. The ergodic distribution Π of X is given by $\Pi(\mathrm{d}x) = \pi(x)\mathrm{d}x$.

Notice that X is completely characterized by (π, s, ϵ). In fact, we can recover b and c by $1/c^2 = \epsilon^2 s'\pi$ and $b = -c^2 s''/2s'$.

Suppose that φ is locally bounded on $\mathbb{R}$ and that there exists a non-empty open set $U \subset \mathbb{R}$ such that on U, φ and ρ are continuously differentiable, $(1-\rho^2)\varphi^2 > 0$ and $|\varphi'| > 0$.

For given $\gamma = (\gamma_+, \gamma_-) \in [0,\infty)^2$ and $\delta \in (0,1)$, denote by $C(\gamma,\delta)$ the set of the triplets $\theta = (\pi, s, \epsilon)$ satisfying the following conditions:

- π is a locally bounded probability density function on $\mathbb{R}$ such that $1/\pi$ is also locally bounded on $\mathbb{R}$,

- s is a bijection from $\mathbb{R}$ to $\mathbb{R}$ such that s' exists and is a positive absolutely continuous function,

- ϵ is a positive finite constant, and...

- It holds that

$$(1+\varphi(x)^2)\pi(x)s'(y) \le \exp\{-\log(\delta) + \gamma_+ x - (4\gamma_+ + \delta)(x-y)\}$$

for all $x \ge y \ge 0$ and

$$(1+\varphi(x)^2)\pi(x)s'(y) \le \exp\{-\log(\delta) - \gamma_- x + (4\gamma_- + \delta)(x-y)\}$$

for all $x \le y \le 0$,

- There exist $x \in U$ and $a \in [\delta, 1/\delta]$ such that $|x| \le 1/\delta$, $[x-a, x+a] \subset U$, π is absolutely continuous on $[x-a, x+a]$ and it holds

$$\sup_{y\in[x-a,x+a]} \left| \left(\sqrt{\frac{\pi}{s'}}\varphi\rho \right)'(y) \right| \vee s'(y) \vee \pi(y) \vee \frac{1}{s'(y)} \vee \frac{1}{\pi(y)} \le 1/\delta.$$

Notice that if $(\pi, s, \epsilon_0) \in C(\gamma, \delta)$, then $(\pi, s, \epsilon) \in C(\gamma, \delta)$ for all $\epsilon > 0$.

Given $\theta \in \mathcal{C}(\gamma,\delta)$, we write $\pi_\theta, s_\theta, \epsilon_\theta, b_\theta, c_\theta, Z^\theta$ for the elements of $\theta = (\pi, s, \epsilon)$, the corresponding coefficients b, c of the stochastic differential equations, and the log price process Z defined as (1) respectively.

Theorem: Fix $\gamma = (\gamma_+, \gamma_-) \in [0,\infty)^2$ and $\delta \in (0,1)$. Denote by $\mathcal{B}_\delta$ the set of the Borel functions bounded by $1/\delta$. Then,

$$\sup_{f\in\mathcal{B}_\delta, \theta\in\mathcal{C}(\gamma,\delta)} \epsilon_\theta^{-2} \left|\mathbb{E}[f(Z_T^\theta)] - \mathbb{E}[(1+p_\theta(N))f(Z_0 - \log(D) - \Sigma_\theta/2 + \sqrt{\Sigma_\theta}N)]\right|$$

is finite, where $N \sim \mathcal{N}(0,1)$, $\Sigma_\theta = \Pi_\theta[\varphi^2]T$, $\Pi_\theta(\mathrm{d}x) = \pi_\theta(x)\mathrm{d}x$ and

$$\begin{aligned} p_\theta(z) &= \alpha_\theta \left\{1 - z^2 + \frac{1}{\sqrt{\Sigma_\theta}}(z^3 - 3z)\right\}, \\ \alpha_\theta &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x} \left\{\frac{\varphi(v)^2}{\Pi_\theta[\varphi^2]} - 1\right\}\Pi_\theta(\mathrm{d}v)\frac{\varphi(x)\rho(x)}{c_\theta(x)}\mathrm{d}x. \end{aligned} \tag{6}$$

Note that

- if $\theta \in \mathcal{C}(\gamma, \delta)$, then $(\pi_\eta, s_\eta, \epsilon_\eta)$ associated with the drift coefficient $b_\eta = b_\theta/\eta^2$ and the diffusion coefficient $c_\eta = c_\theta/\eta$ is also an element of $\mathcal{C}(\gamma, \delta)$ for any $\eta > 0$.

- This is because $\pi_\eta = \pi_\theta$ and $s_\eta = s_\theta$.

- On the other hand, $\epsilon_\eta = \eta\epsilon_\theta$,

so that our main theorem implies, with a slight abuse of notation,

$$E[f(Z_T^\eta)] = E[(1 + p_\eta(N))f(Z_0 - \log(D) - \Sigma_\eta/2 + \sqrt{\Sigma_\eta}N)] + O(\eta^2) \qquad (7)$$

as $\eta \to 0$.