CIID default models and implied copulas

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Aims and Agenda

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> 1.) **Present a unified framework for CIID models**

2.) Axiomatically define desirable statistical properties

3.) Review classical models

4.) Present new models

Motivation: CDO pricing

- **Situation:**
  - > Portfolio of d credit-risky assets, $(\tau_1, \ldots, \tau_d)'$ vector of random default times.
  - > $L_t := \frac{1}{d} \sum_{k=1}^{d} \mathbf{1}_{\{\tau_k < t\}}$, i.e. percentage of defaults up to time $t \geq 0$.

- **Two major problems:**
  (1) Typically $d = 125$, i.e. large.
  (2) Default times are dependent.

- **Pricing of CDOs without simulation:**
  - > Assumptions: Constant and identical recovery rates, equal portfolio weights.
  - > Pricing of CDOs requires:

$$\mathbb{E}\big[f(L_t)\big] = \int_{[0,1]} f(x)\,\mathbb{P}(L_t \in dx), \quad f \text{ complicated (collar-type).}$$

Modeling $(\tau_1, \ldots, \tau_d)'$: Model philosophies

- **Structural models:**
  i.) Model correlated asset processes, ii.) determine $(\tau_1, \ldots, \tau_d)'$, and iii.) $\{L_t\}_{t \geq 0}$.

  ☺: Economic interpretation of (correlated) defaults.

  ☹: Distribution $\mathbb{P}(L_t \in dt)$ very difficult to obtain.

- **Bottom-Up models:** i.) Model $(\tau_1, \ldots, \tau_d)'$, ii.) compute $\{L_t\}_{t \geq 0}$.

  ☺: (Intuitive) model for dependence between firms.

  ☹: Distribution $\mathbb{P}(L_t \in dt)$ difficult to obtain.

- **Top-Down models:** i.) Model $\{L_t\}_{t \geq 0}$ directly.

  ☺: Distribution $\mathbb{P}(L_t \in dt)$ tractable.

  ☹: Dependence structure between firms?

CIID default models

- **Definition (CIID model):**

$$\tau_k := \text{function}(M, \epsilon_k),$$

  > M is a random object (market risk factor),

  > $\epsilon_1, \ldots, \epsilon_d$ are i.i.d. and independent of M (idiosyncratic risk factors).

- **Consequences (simple Bottom-Up model, dependence via M):**

  ☹: Restrictive assumptions, e.g.

  > All default times have the same distribution $\mathbb{P}(\tau_k \leq t) =: F(t)$.

  > The dependence structure is „very special": conditionally i.i.d. (CIID).

  ☺: Large portfolio assumption:

  > The model approximates a related Top-Down model.

  > Closed-form approximation of portfolio-loss distribution / CDO prices.

CIID models: general framework

Lemma (Unified framework):

- All CIID models can be constructed as follows:

  (1) Let $\{F_t\}_{t\geq 0}$ be càdlàg, $\nearrow$, with $F_0 = 0$, and $\lim_{t\to\infty} F_t = 1$ $(\forall\, \omega \in \Omega)$.

  (2) Given $\sigma(F_t\,:\,t \geq 0)$, let $\tau_1, \ldots, \tau_d$ be i.i.d. with cdf $t \mapsto F_t$.

Lemma (Canonical construction):

- Define $(\tau_1, \ldots, \tau_d)'$ via:

$$\tau_k := \inf\{t > 0\,:\,F_t \geq U_k\},$$

  where $U_1, \ldots, U_d \sim \mathrm{Uni}(0,1)$ are i.i.d. and independent of $\{F_t\}_{t\geq 0}$.

Consequence:

- A CIID model is basically a model for the **market frailty** $\{F_t\}_{t\geq 0}$.

Large homogeneous portfolio approximation

Lemma (Portfolio-loss distribution):

- The distribution of the portfolio loss is available but numerically critical:

$$\mathbb{P}\Big(L_t = \frac{k}{d}\Big) = \binom{d}{k}\,\mathbb{E}\big[F_t^k\,(1-F_t)^{d-k}\big], \quad k = 0, 1, \ldots, d.$$

- Gliwenko-Cantelli:

$$\mathbb{P}\Big(\lim_{d\to\infty}\,\sup_{t\geq 0}\big|F_t - L_t\big| = 0\Big) = 1.$$

- For $d \gg 2$, this justifies:

$$\mathbb{E}[f(L_t)] = \int_{[0,1]} f(x)\,\mathbb{P}(L_t \in dx) \approx \int_{[0,1]} f(x)\,\mathbb{P}(F_t \in dx).$$

- Equivalent to using a Top-Down model with $L_t := F_t$.

Examples

- Model input: Term structure of default probabilities $t \mapsto F(t) := \mathbb{P}(\tau_1 \leq t)$.

1.) **Gaussian copula model:** [Vasicek 1987, Li 2000]

$$\tau_k := F^{-1}\Big(\Phi\big(\sqrt{\rho}\,M + \sqrt{1-\rho}\,\epsilon_k\big)\Big)$$

$$F_t := \Phi\Big(\frac{\Phi^{-1}\big(F(t)\big) - \sqrt{\rho}\,M}{\sqrt{1-\rho}}\Big), \quad t \geq 0,$$

where $M, \epsilon_1, \ldots, \epsilon_d$ are i.i.d. standard normal.

2.) Generalization to **infinitely divisible distributions:** [Albrecher et al. 2007]

$$F_t := H_{[1-\rho]}\Big(H_{[1]}^{-1}\big(F(t)\big) - M\Big), \quad t \geq 0,$$

where $H_{[t]}$ = cdf of X_t for some suitable Lévy process $\{X_t\}_{t \in [0,1]}$ and $M := X_\rho$.

> Corresponds to replacing normal distribution by ID distribution.

> Special cases: [Guégan, Houdain 2005], [Kalemanova et al. 2007], ...

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Desirable properties
(Sep) and (Cop)

Definition (Sep), (Cop):

- (Sep) $:\Leftrightarrow \mathbb{E}[F_t] = F(t) = \mathbb{P}(\tau_k \leq t)$ for all $t > 0$.

  Advantage: The marginal distribution $t \mapsto F(t)$ is model input.

- (Cop) $:\Leftrightarrow \exists$ explicit expression for:

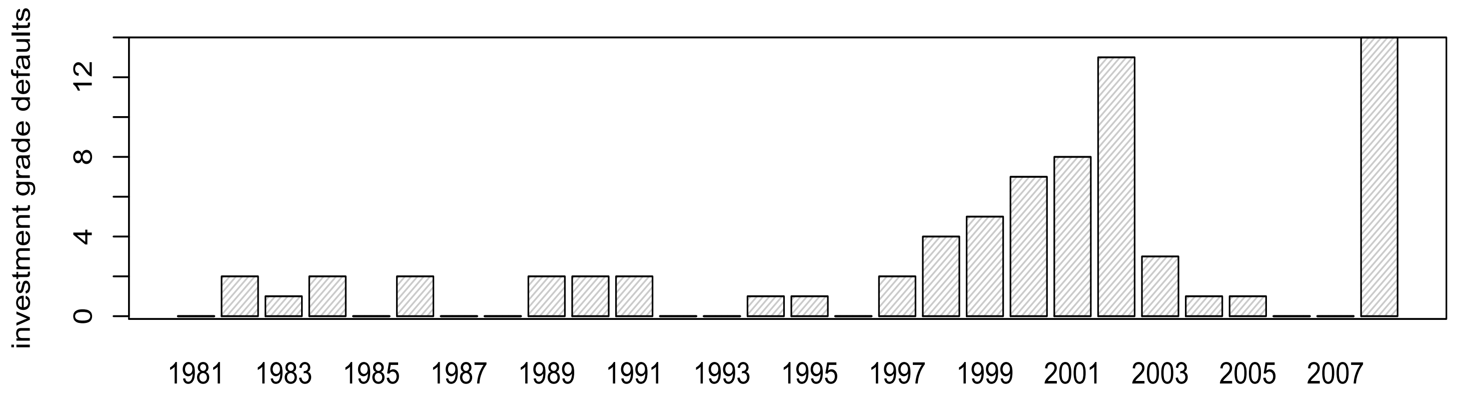
$$\mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = \mathbb{E}\big[F_{t_1} \cdots F_{t_d}\big], \quad t_1, \ldots, t_d \geq 0.$$

- If both (Sep) and (Cop) hold, then one finds the (survival) copula:

$$C(u_1, \ldots, u_d) = \mathbb{E}\big[F_{F^{-1}(u_1)} \cdots F_{F^{-1}(u_d)}\big], \quad u_1, \ldots, u_d \in [0,1],$$
$$\hat{C}(u_1, \ldots, u_d) = \mathbb{E}\big[(1 - F_{F^{-1}(1-u_1)}) \cdots (1 - F_{F^{-1}(1-u_d)})\big], \quad u_1, \ldots, u_d \in [0,1].$$

- **Copula models:** dependence structure $\oplus$ marginal default probabilities.

Desirable properties
(Exc)



- **Stylized facts:** Excess clustering, multiple defaults.
  - > Excess clustering $\leftrightarrow$ „fast growth" of $\{F_t\}_{t\geq 0}$.
  - > Multiple defaults $\leftrightarrow$ jumps of $\{F_t\}_{t\geq 0}$ $\leftrightarrow$ singular component of $C, \hat{C}$.
- **Definition (Exc):**
  (Exc) $:\Leftrightarrow$ $\{F_t\}_{t\geq 0}$ exhibits jumps $\Leftrightarrow$ $\mathbb{P}(\tau_1 = \ldots = \tau_k) > 0$.

Desirable properties
(Fs)

Definition (Fs):

- **($Fs_{\ominus}$): Static source of frailty**: F_t is $\bigcap_{u>0} \sigma(F_s : 0 \leq s \leq u)$-measurable.
  > E.g. $F_t :=$ function(M, t), where M is a random parameter.

  > Unintuitive model, since no time-evolution.

  > Typically $t \mapsto F_t$ smooth function (no jumps, no time-varying slopes).
- **($Fs_{\odot}$):** Dynamic frailty with **time-homogeneous innovations**.
  > E.g. $\{F_t\}_{t \geq 0}$ driven by Lévy process (no stoch. vol.).
- **($Fs_{\oplus}$):** Dynamic frailty with **time-inhomogeneous innovations**.

Desirable properties
(Tdc)

- **Coefficient of lower-tail dependence:**

$$\lambda_l := \lim_{t \downarrow 0} \mathbb{P}(\tau_i \leq t \,|\, \tau_j \leq t) = \lim_{t \downarrow 0} \frac{\mathbb{E}\big[F_t^2\big]}{\mathbb{E}[F_t]}.$$

  > Measures the likelihood of joint early defaults.

  > Empirical studies suggest that (Tdc)-supporting models are more successful in explaining CDO quotes.

  > **Attention**: Only bivariate margins considers, higher-order effects neglected!

- **Definition (Tdc):**
  (Tdc) $:\Leftrightarrow \lambda_l > 0$.

Desirable properties
(Den)

- **Density of the portfolio-loss approximation:**
  - > Implementation requires viable distribution of F_t (for all $t > 0$).
  - > Sometimes, the density is only available through Laplace-inversion techniques.

- **Definition (Den):**
  (Den) $:\Leftrightarrow$ the density of F_t is known explicitly for all $t > 0$.

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Examples - cont.

- Model input: Term structure of default probabilities $t \mapsto F(t) := \mathbb{P}(\tau_1 \leq t)$.

3.) **Archimedean copula model:** [Schönbucher 2002]

$$\tau_k := \inf\Big\{t \geq 0 : U_k \leq 1 - \exp\big(- M\varphi^{-1}(1 - F(t))\big)\Big\},$$
$$F_t := 1 - \exp\Big(- M\,\varphi^{-1}\big(1 - F(t)\big)\Big), \quad t \geq 0,$$

$M > 0$ a positive random variable with Laplace transform φ, $U_k \sim \mathrm{Uni}(0,1)$.

4.) **Lévy-frailty model:** [Mai, Scherer 2009]

$$\tau_k := \inf\{t \geq 0 : M_t \geq E_k\}, \quad E_k \sim \mathrm{Exp}(1),$$
$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{-\log(1-F(t))/\Psi(1)}, \quad t \geq 0,$$

with $\Lambda = \{\Lambda_t\}_{t\geq 0}$ a Lévy subordinator with LE $\Psi(x) = -\log \mathbb{E}[\exp(-x\Lambda_1)]$.

Examples - cont.

5.) **Intensity based model:** [Duffie, Gârleanu 2001]

> In their general form not CIID models, but in the following special case:

$$\tau_k := \inf\{t \geq 0 : M_t \geq E_k\}, \quad E_k \sim \text{Exp}(1),$$
$$F_t := 1 - e^{-M_t}, \quad M_t := \int_0^t \lambda_s \, ds, \quad t \geq 0,$$

where $\{\lambda_t\}_{t \geq 0}$ is a basic affine process, i.e.

$$d\lambda_t = \kappa\,(\theta - \lambda_t)\,dt + \sigma\,\sqrt{\lambda_t}\,dB_t + dZ_t, \quad \lambda_0 > 0.$$

> Default probabilities are not model input. However,

$$F(t) = 1 - \mathbb{E}\big[e^{-M_t}\big] = 1 - \exp(\alpha(1,t) + \beta(1,t)\,\lambda_0),$$

for explicit functions $t \mapsto \alpha(1,t)$, $\beta(1,t)$, see [Duffie, Kan 1996].

Properties of the models

Model	(Sep)	(Cop)	(Exc)	(Fs)	(Tdc)	(Den)
1.) Gaussian	☺	Gaussian	☹	☹	☹	☺
2.) ID-Extension	☺	?	☹	☹	☺	☺
3.) Archimedean	☺	Archimedean	☹	☹	☺	☺
4.) Lévy-frailty	☺	Marshall-Olkin	☺	☹ ☺	☺	☺
5.) Intensity	☹	?	☹	☺	☹	☺

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A new model based on Archimax copulas

Combining [Schönbucher 2002] and [Mai, Scherer 2009]:

- Define

$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{\bar{M}\varphi^{-1}(1-F(t))/\Psi(1)}, \quad t \geq 0,$$

where Λ as before and $\bar{M} > 0$ with Laplace trafo φ, independent of Λ.

> Combination of Archimedean and Marshall-Olkin copulas ($\subset$ Archimax copulas).

☺ Inherits benefits from / extends earlier approaches.

☹ (Den) is lost (only known for special cases, Laplace-inversion required).

A new model based on Archimax copulas

Properties:

- It can be shown that $(\tau_1, \ldots, \tau_d)' \sim \hat{C}\big(F, \ldots, F\big)$ with

$$\hat{C}(u_1, \ldots, u_d) = \varphi\Big(\frac{1}{\Psi(1)} \sum_{i=1}^{d} \varphi^{-1}(u_{(i)}) \big(\Psi(i) - \Psi(i-1)\big)\Big),$$

where $u_{(1)} \leq \ldots \leq u_{(d)}$ denotes the ordered list of $u_1, \ldots, u_d \in [0, 1]$.

Special case (Clayton mixed with Cuadras-Augé):

> Λ Poisson process with intensity $\beta > 0$.

> $\bar{M} \sim \Gamma(1, 1/\theta)$.

> Then, it can be shown that

$$\mathbb{P}(\Lambda_{\bar{M}\,t} = k) = \frac{(t\,\beta)^k}{\Gamma(1/\theta)\,k!} \Big(\frac{1}{1+\beta\,t}\Big)^{k+\frac{1}{\theta}} \Gamma\Big(k + \frac{1}{\theta}\Big), \quad k \in \mathbb{N}_0.$$

A new model based on a CGMY-type frailty

Combining [Duffie, Gârleanu 2001] and [Mai, Scherer 2009]:

- Define

$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{\int_0^t \lambda_s\, ds/\Psi(1)}, \quad t \geq 0,$$

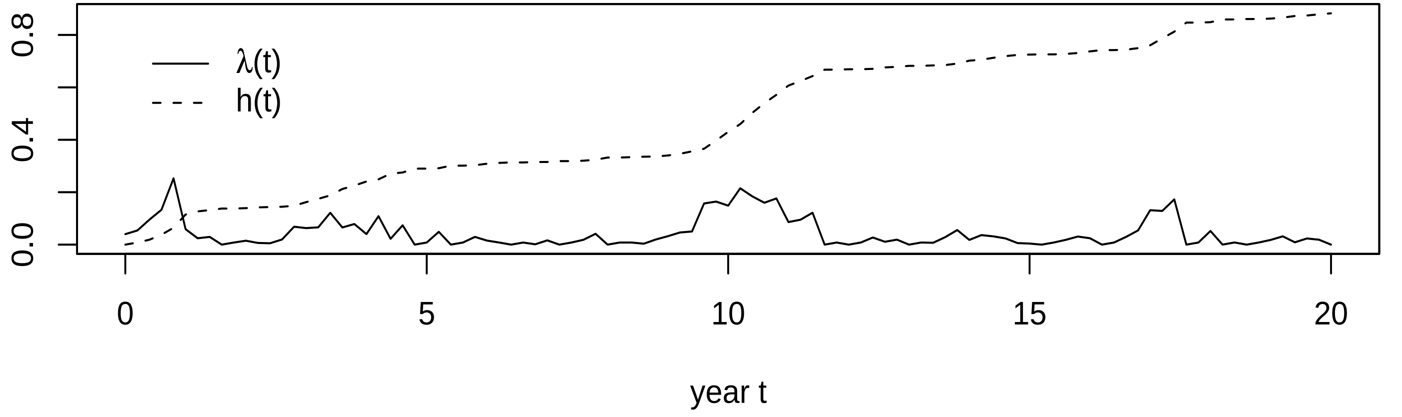
where Λ and λ are given as before.

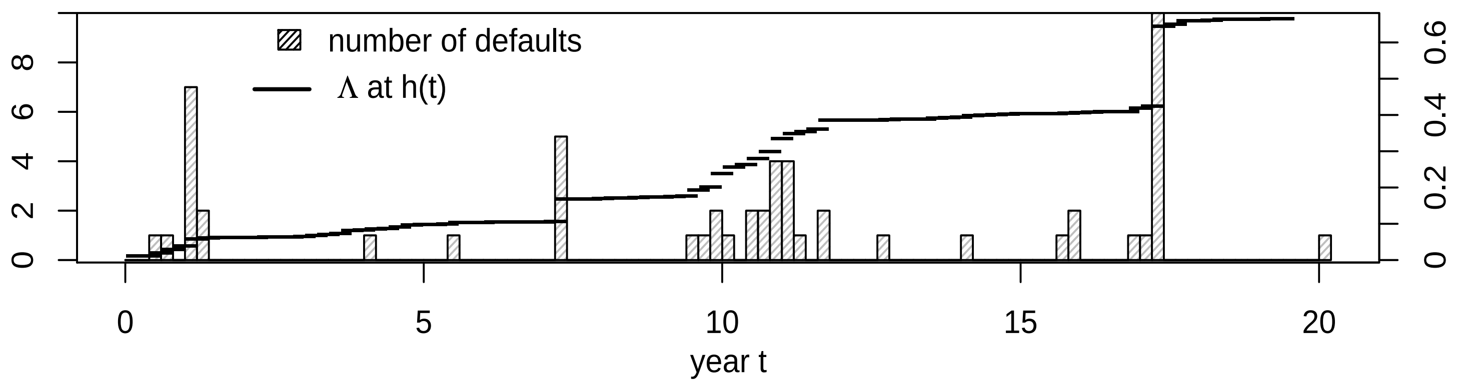
\> Time-varying intensity + jumps (see stoch.vol. + jumps in asset models).

☺ Incorporates most stylized facts, intuitive and flexible model.

☹ (Den) is lost (Laplace-inversion required), (Cop) is lost, (Sep) „partially“ lost.

A new model based on a CGMY-type frailty





A new model based on a CGMY-type frailty

Properties:

- Lévy subordinator Λ accounts for clustering, e.g.

$$\mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{\sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i)}{\Psi(k)}, \quad k = 2, \ldots, d.$$

- Lévy subordinator Λ also accounts for tail-dependence:

$$\lambda_l = 2 - \Psi(2)/\Psi(1).$$

- Intensity λ accounts for time-inhomogeneous distribution of the clusters.

- Required density must be obtained from the following Laplace trafo:

$$\mathbb{E}\Big[\exp\Big(-q\,\Lambda_{\int_0^t \lambda_s\,ds\,/\Psi(1)}\Big)\Big] = e^{\alpha\left(\Psi(q)/\Psi(1),t\right)+\beta\left(\Psi(q)/\Psi(1),t\right)\lambda_0}, \quad q \geq 0,$$

where $\alpha(z,t)$, $\beta(z,t)$ are known in closed form from [Duffie, Kan (96)].

Properties of the models

Model	(Sep)	(Cop)	(Exc)	(Fs)	(Tdc)	(Den)
1.) Gaussian	☺	Gaussian	☹	☹	☹	☺
2.) ID-Extension	☺	?	☹	☹	☺	☺
3.) Archimedean	☺	Archimedean	☹	☹	☺	☺
4.) Lévy-frailty	☺	Marshall-Olkin	☺	☹☺	☺	☺
5.) Intensity	☹	?	☹	☺	☹	☺
3.) & 4.)	☺	Archimax	☺	☹☺	☺	☹☺
4.) & 5.)	☹☺	?	☺	☺	☺	☹

Conclusion

Conclusion

- A unified framework and a canonical construction for CIID models is given.
- Desirable statistical properties are identified and axiomatically defined.
- Existing models are analyzed in this regard.
- Two new models are presented.

References

Thank you for your attention.

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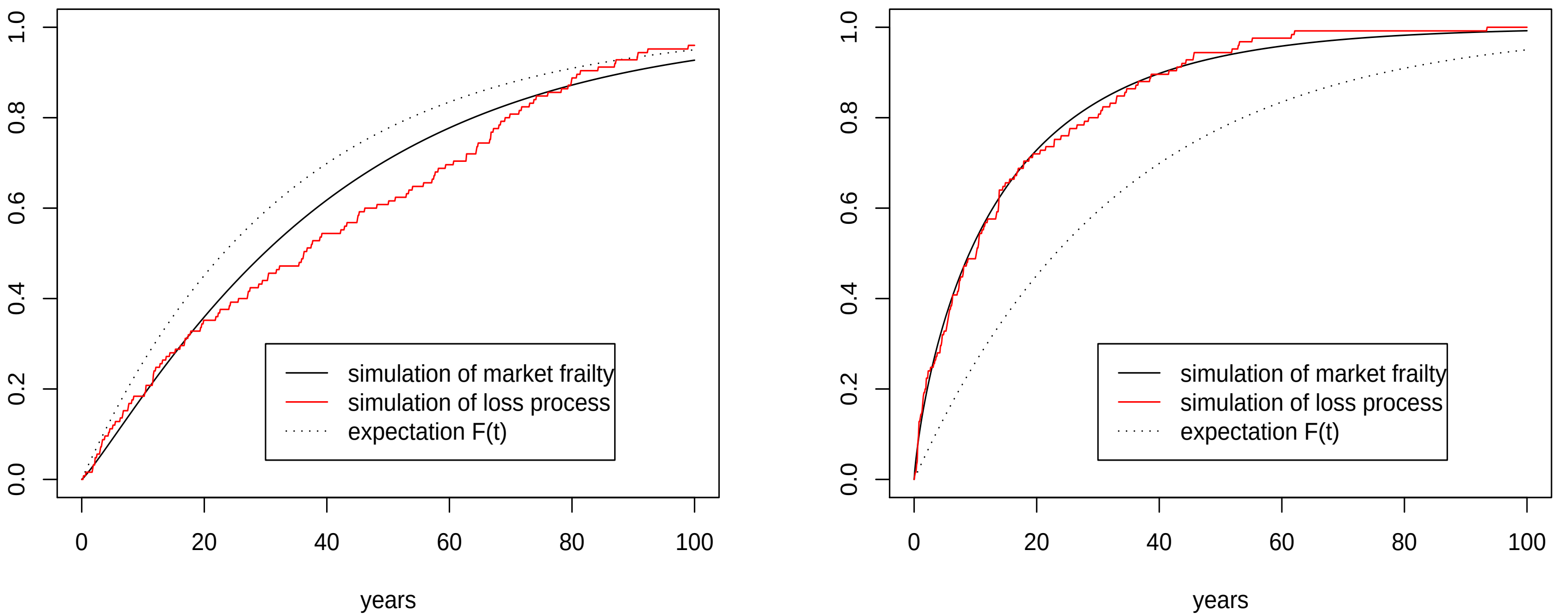
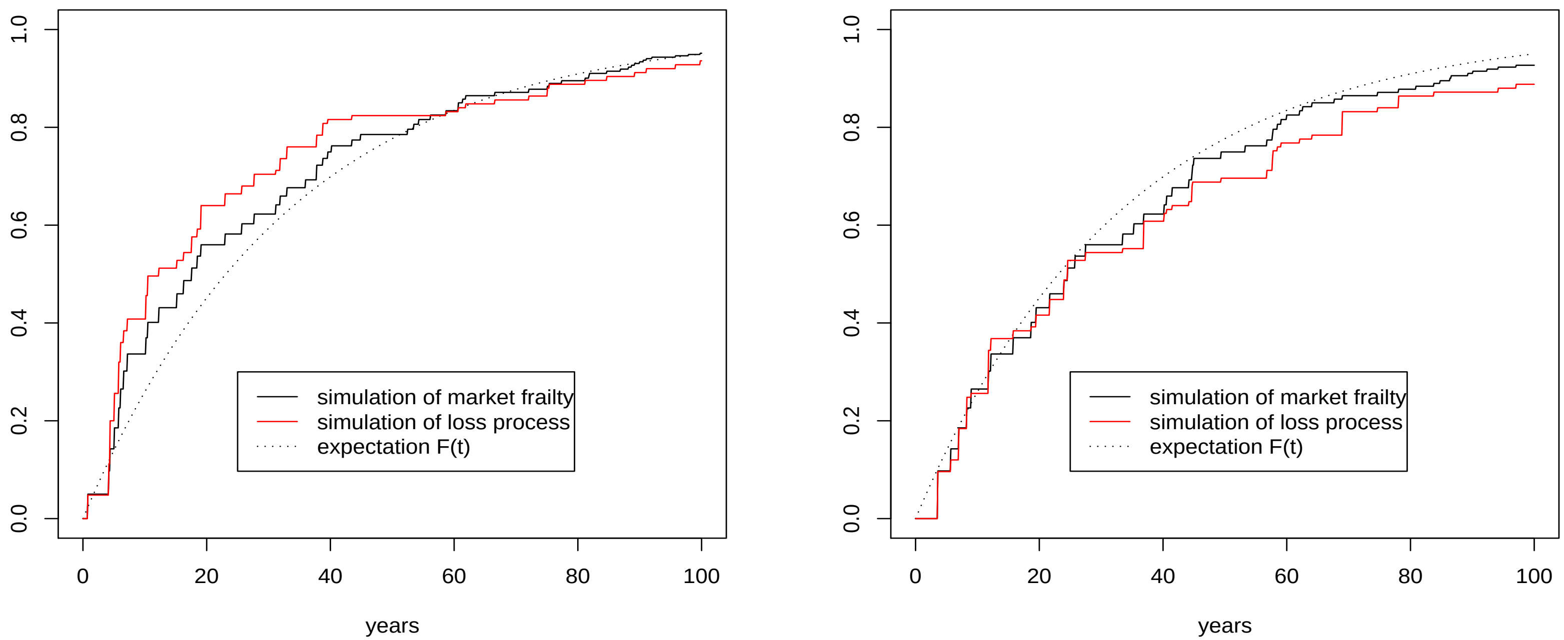
Example: Gaussian copula model [Li 2000]



$$F_t = \Phi\Big(\frac{\Phi^{-1}\big(F(t)\big) - \sqrt{\rho}\, M}{\sqrt{1-\rho}}\Big), \quad F(t) = \mathbb{E}[F_t] = 1 - \exp(-0.03\, t)$$

A Lévy-based model [Mai, Scherer 2009]



$$F_t = 1 - (1-\rho)^{N_{0.03\,t}}, \quad N_1 \sim \mathrm{Poi}(1/\rho), \quad F(t) = \mathbb{E}[F_t] = 1 - \exp(-0.03\,t)$$

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