Optimized Least-squares Monte Carlo (OLSM) for Measuring Counterparty Credit Exposure of American-style Options

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1. Introduction
   - Least-squares Monte Carlo (LSM)
   - Exposure Estimation by OLSM
2. OLSM
   - Convergence Speedup
   - Improving the fitness of the regression
3. Concluding Remarks and Future Directions

Counterparty Exposure

- "Counterparty exposure is the larger of zero and the market value of an option that would be lost to the counterparty if the counterparty were to default and there were zero recovery."
- Market value of an option = (Risk-neutral) Option value corresponding to the real-world value of the underlying risk factors.

Nested Simulations

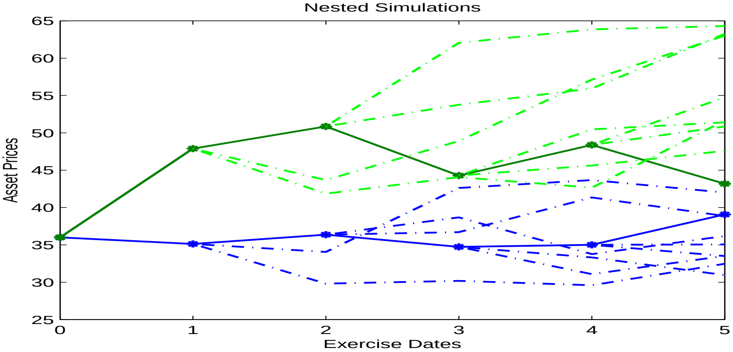

Figure: Nested simulations for estimating counterparty exposures.

American Option Pricing

Definition: A contract that can be exercised at any time up to and including the expiration date at a specified strike price.

Value at time k:

$$B_k = \max_{\tau \in [k,\ldots,N]} \mathbb{E}[e^{-r\tau} P_\tau | \mathcal{F}_k]$$

Recursive equations:

$$H_k = \mathbb{E}[e^{-r\Delta T} B_{k+1} | \mathcal{F}_k]$$

$$B_k = \max(H_k, P_k)$$

where

- $\tau \in [k, \ldots, N]$ is the stopping time;
- P_k is the option payoff;
- H_k is the continuation value, $H_N = 0$;
- B_k is the option value.

Continuation Value Estimation

- Simulate M sample paths to option maturity.
- Estimate the continuation value by cross-sectional linear regression.

$$e^{-r\Delta T}\tilde{B}^i_{k+1} = x^i_k\beta_k + \epsilon^i_k, \text{ i} = 1,2,\ldots,\text{M},$$

where

- ϵ^i_k is a noice term;
- $\tilde{B}^i_{k+1}$ is the option value estimator;
- x^i_k is a known row vector of basis functions;
- β_k is a column vector of regression coefficients.

Continuation value estimator

$$\tilde{H}^i_k = x^i_k\tilde{\beta}_k = x^i_k(X'X)^{-1}X'\tilde{B}$$

LSM Estimators

- $\tilde{H}_k^i$ are used to make exercise decisions.
- If the option is not exercised, the continuation values are the discounted option values / cash flows.
- LSM estimators:

$$\tilde{H}_k^i = x_k^i \tilde{\beta}_k$$

$$\hat{H}_k^i = e^{-r\Delta T} \tilde{B}_{k+1}^i$$

$$\tilde{B}_k^i = \begin{cases} \hat{H}_k^i & \text{if } \tilde{H}_k^i > P_k^i \\ P_k^i & \text{if } \tilde{H}_k^i \leq P_k^i \end{cases}$$

where $\tilde{H}_N^i = \hat{H}_N^i = 0$.

OLSM Framework

1. Simulate underlying stock prices under the risk-neutral measure.
2. Perform LSM on these risk-neutral stock prices.
   — Estimated continuation value function (CVF) obtained at every exercise opportunity.
3. Simulate underlying stock prices under the real-world measure.
4. Plug the real-world stock prices into the CVF to get the continuation values.
5. Exposure is the maximum of the continuation value and the exercise value. Future exposures are set to zero after the exercise date.

Benchmark Instrument

American call option on a single stock with no dividend:

- Time to maturity: $T = 2$ (years)
- Strike price: $K = 40$
- Initial stock price: $S_0 = 36$
- Risk-free rate: $r = 6\%$ (annual, flat)
- Volatility: $\sigma = 40\%$ (annual)
- Real drift: $\mu = 20\%$ (annual)

Reasons:

- Exists an analytical solution for this option
- Optimal stopping time is the maturity date

Simulation Setup

- 10,000 sample paths (generated externally in practice)
- 40 equidistant time steps over 2 years
- Underlying asset prices follow Geometric Brownian motion (GBM) under risk-neutral (for estimating continuation value functions) and real-world (for calculating exposures) measures, respectively
- Basis functions used in regression are monomials up to the 3^{rd} degree
- 20 independent replications

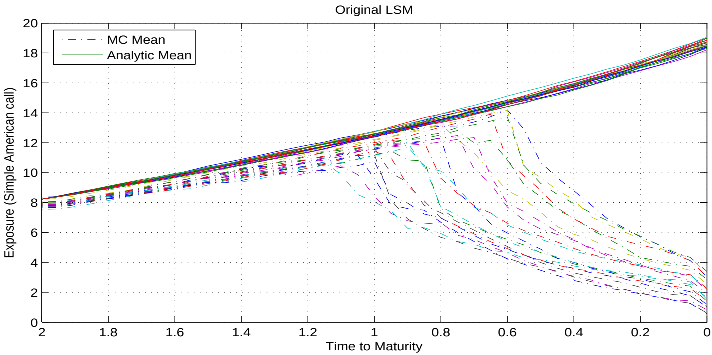

Figure: Exposure versus Time-to-Maturity. 20 Monte Carlo and 20 analytic mean exposures.

Variance Reduction

- Antithetic variates: Applied to the 'exposure' paths.
- Inner control variates: Applied to the response variables in the regression at each exercise opportunity. The control variates are martingales related to monomials up to the 3^{rd} degree.
- Both techniques are effective, but aren't helpful in reducing regression error.

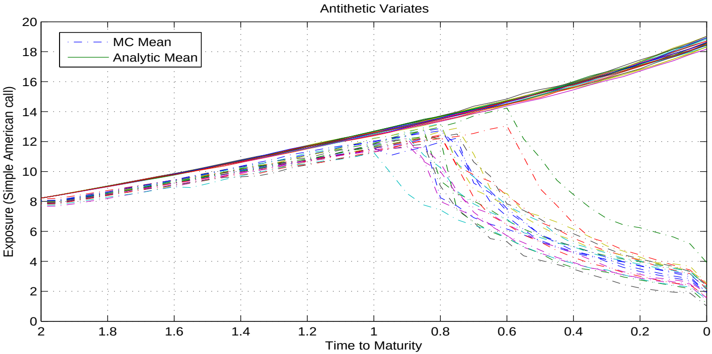

Figure: Antithetic variates are used on the 'exposure' paths.

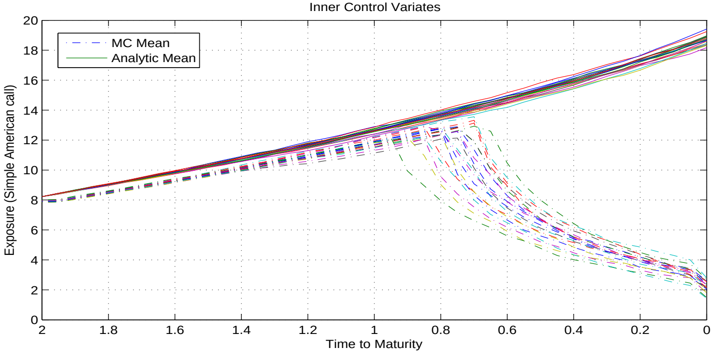

Figure: Inner control variates are used in estimating continuation value functions.

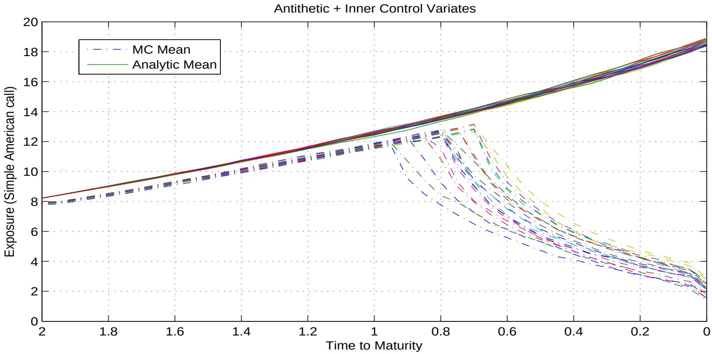

Figure: Both antithetic and inner control variates are used.

Multiple Bucketing

- Multiple Bucketing = Piecewise Linear Regression
- The continuation value function is smoother at the beginning and is less smooth near the maturity date.
- Implications: Use 'one' bucket at the beginning, two buckets thereafter.

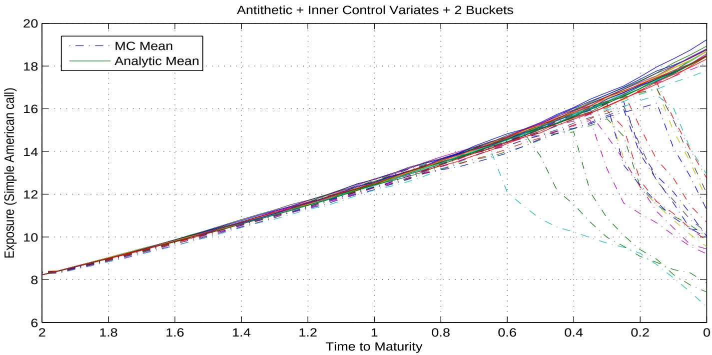

Figure: Two buckets are used, where the bucket boundary is the strike price.

Comments

- The use of the in-the-money (ITM) and out-of-the-money (OTM) buckets significantly improves the convergence of the exposures.
- Inaccurate exposures near the maturity are attributed to the extrapolation error in the regression for large stock prices.
- Initial state dispersion can help avoid this error.

Initial State Dispersion

- Allocate the initial states using the ratio 4:2:4 to (10,80),(80,300) and (300,460); the initial states are equally-spaced within each region
- Regions chosen based on the distribution of the underlying asset prices under risk-neutral and real-world measures

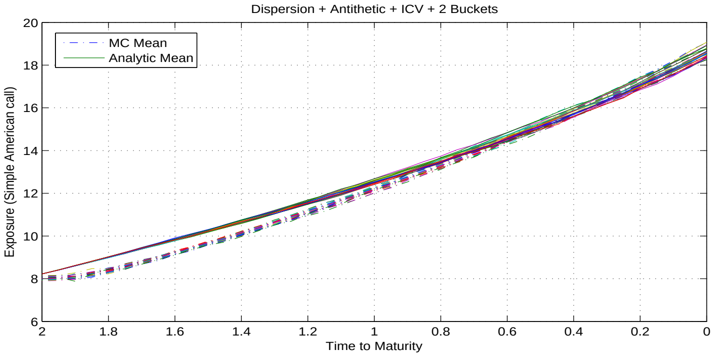

Figure: Two buckets are used, where the bucket boundary is the strike price. Initial states are dispersed based on the ratio 4:2:4 to (10,80),(80,300) and (300,460).

Comments

- Benefits: More accurate exposures near the maturity
- Drawbacks: Larger errors in the exposures in the short term.
- Solution: Use a larger bucket at the beginning.

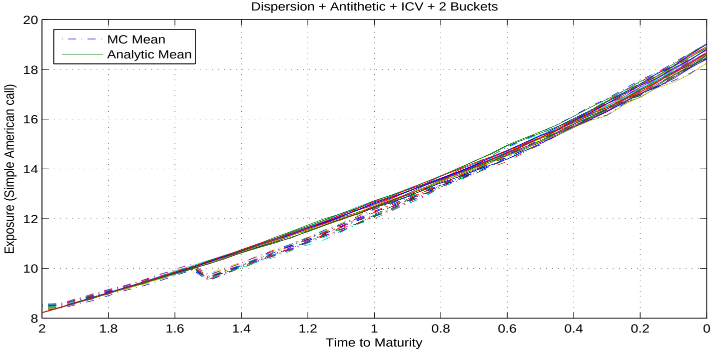

Figure: Two buckets are used, where the bucket boundary is 100 for the first quarter of the option's life, and is the strike price thereafter. Initial states are dispersed based on the ratio 4:2:4 to (10,80),(80,300) and (300,460).

Caution!

- The success of using 'one' bucket in the first quarter tempts us to use it for a longer period, say, half of the maturity. However, that does not necessarily give a better result as the smoothness of the continuation-value curve fades with time.

97.5% Exposure Quantiles

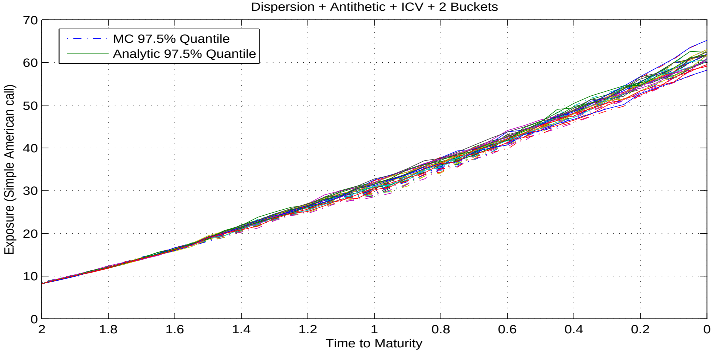

Figure: Two buckets are used, where the bucket boundary is 100 for the first quarter of the option's life, and is the strike price thereafter. Initial states are dispersed based on the ratio 4:2:4 to (10,80),(80,300) and (300,460).

Comments

- For the first quarter, apparently, the 97.5% quantiles of the simulated exposures are perfect!
- The errors of the 97.5% quantiles are well within 10% near the maturity.

OLSM in a Nutshell

- Antithetic variates for 'exposure' paths.
- Inner control variates for 'estimation' paths.
- Two buckets, where the bucket boundaries are [100,40,40,40] for the four quarters, respectively.
- Initial states dispersed based on the ratio 4:2:4 to (10,80),(80,300) and (300,460).

Concluding Remarks

- OLSM generates more reasonable exposures over the option's whole life
- Multiple bucketing and initial state dispersion significantly improve the accuracy of the estimated exposures near the maturity.
- OLSM is easily applicable to higher dimensional problems that might involve more complex payoff functions, stochastic interest rates and stochastic volatility processes for underlying risk factors

Future Directions

- Develop a systematic method to pick the bucket boundaries.
- Explore other ways to disperse the initial states that would reduce exposure variance, while maintaining the same accuracy.

Thank you for your attention!

Questions or comments?