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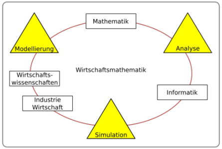
Hedging under Model Uncertainty

Efficient Computation of the Hedging Error using the POD

6th World Congress of the Bachelier Finance Society
June, 24th 2010



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Hedging under Model Uncertainty

Considered Models

Reduced Model and POD

Results

Hedging a exotic Option under Model Uncertainty

Exotic Option: Asian Call

$$C^A(T, S_T) = \left(\bar{S}_T - K\right)^+$$

- Model uncertainty: “true model” $\neq$ hedge model
- Relation between “true model” and hedge model:
  - Vanilla Options for the calibration of the hedge model parameters

Hedging a exotic Option under Model Uncertainty

Exotic Option: Asian Call

$$C^A(T, S_T) = \left(\bar{S}_T - K\right)^+$$

- Model uncertainty: “true model” $\neq$ hedge model
- Relation between “true model” and hedge model:
  - Vanilla Options for the calibration of the hedge model parameters
- Hedging Approach:
  - Delta-Hedge with bank account and underlying
  - Delta- and Vega-Hedge with bank account, underlying and vanilla option

“True Models”

3-Factor Model (3FM)

(S_t, v_t, ρ_t) driven by (W_t^1, W_t^2, W_t^3)

Extended Heston Model with correlated jumps (SVJJ)

$S_t = S_0 e^{x_t}$ where

$$dx_t = \left(\mu - \frac{1}{2} v_t\right) dt + \sqrt{v_{t-}} dW_t^1 + \xi^x dN_t$$

$$dv_t = \alpha (\beta - v_t) dt + \sigma_v \sqrt{v_{t-}} \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2\right) + \xi^v dN_t$$

Extended CGMY Model (CGMYe)

$$S_t = S_0 \exp\left\{(\mu + \omega - \eta^2/2)\, t + X_t^{\mathrm{CGMY}} + \eta W_t\right\}$$

Hedge Models

Black-Scholes Model (BS)

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Stochastic Alpha, Beta, Rho Model (SABR)

$$dS_t = rS_t dt + \sigma_t S_t dW_t^1$$

$$d\sigma_t = \alpha\sigma_t \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right)$$

Heston Model (SV)

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1$$

$$dv_t = \alpha(\beta - v_t)\, dt + \sigma_v \sqrt{v_t} \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right)$$

Model Properties

“True Models”

- Models represent various “stylized facts”.
- Calibrated parameter are available for $\mathbb{P}$ and for $\mathbb{Q}$.
- Vanilla Option prices available via
  - Monte-Carlo (3FM)
  - Analytic Formula (SVJJ)
  - PIDE (CGMYe)
- Valuation of the Asian Option by Monte-Carlo.

Hedge Models

- (Semi-)analytic formulas available for the Vanilla Option prices.
- Valuation of the Asian Option and hedging weights via PDE.

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PDE in Heston Model

Idea: Vecer ’02, Shreve ’08 $\Rightarrow C^A(t, S_t, v_t) = S_t g(t, Y_t, v_t)$ where

2-dimensional parabolic PDE in Heston Model

$$
\begin{aligned}
\frac{\partial}{\partial t} g(t,y,v) + \tilde{\nu}\left(\tilde{\phi} - v\right)\frac{\partial}{\partial v} g(t,y,v) + \frac{1}{2} v\,(q_t - y)^2\,\frac{\partial^2}{\partial y^2} g(t,y,v) \\
+ \frac{1}{2}\varphi^2 v \frac{\partial^2}{\partial v^2} g(t,y,v) + \varphi\rho v\,(q_t - y)\,\frac{\partial^2}{\partial y \partial v} g(t,y,v) = 0, \\
g(T,y,v) = y^+
\end{aligned}
$$

and

$$
Y_t = \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) \qquad \text{on } \Omega = (-1,1)\times(0,\infty).
$$

Further Advantage: Greeks are directly computable from the PDE.

PDE in Black-Scholes Model

Again we have $C^A(t, S_t) = S_t g(t, Y_t)$ where this time

1-dimensional parabolic PDE in Black-Scholes Model

$$\frac{\partial}{\partial t} g(t,y) - \frac{\sigma^2}{2}(q_t - y)^2 \frac{\partial^2}{\partial y^2} g(t,y) = 0,$$
$$g(0,y) = y^+ \qquad \text{on } \Omega = (-1,1)$$

Similar PDE for Λ (Vega)

$$\frac{\partial}{\partial t}\Lambda(t,y) - \frac{\sigma^2}{2}(q_t - y)^2 \frac{\partial^2}{\partial y^2}\Lambda(t,y) = \sigma (q_t - y)^2 \frac{\partial^2}{\partial y^2} g(t,y)$$

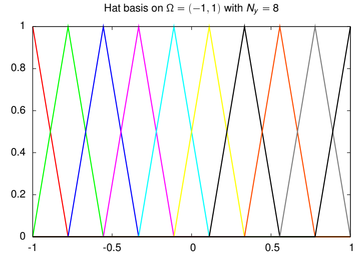
Reduced Basis Methods and POD

Problem

Many solutions of the PDE are necessary for the Calculation of the Hedge weights in the simulation ($N \cdot 63$), each with different parameters.

$\Longrightarrow$ Approximate the solution with a reduced model.

Instead of the classical hat basis (FE)

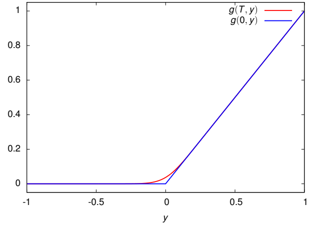


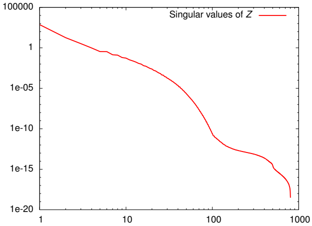
try to use a reduced basis consisting of “empirical” eigenfunctions.

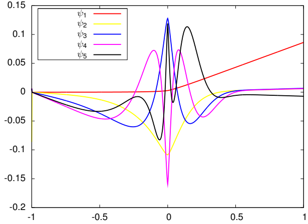
Reduced Basis for the Black-Scholes Model

Store snapshots of the solution $g(t_i, y_j; \theta)_{i,j=1}^{M,N}$ of the PDE in a matrix $Z = (g(t_i, y_j; \theta_\ell))_{i,j,\ell=1}^{M,N,\ell}$ and calculate via the SVD the reduced basis.

Example: Calculation of $g(t, y)$ with $N_y = 801, M = 400$







Offline: Calculate the reduced basis.
Online: Use $\mathcal{N} \ll N$ degrees of freedom for the actual computation.

Reduced Basis – Efficiency in Black-Scholes Model

FEM-calculation with 801 basis functions: 35 seconds

σ	$C_A(0, 100)$	Delta	Vega
0.1	10.5969	0.989909	0.32507
0.188889	10.8205	0.926435	5.15053
0.277778	11.4764	0.845893	9.20365
0.366667	12.4004	0.787475	11.3734
0.455556	13.4674	0.747672	12.5557
0.544444	14.6143	0.720255	13.2377
0.633333	15.8083	0.701023	13.6474
0.722222	17.0308	0.687362	13.8967
0.811111	18.2704	0.677634	14.0426
0.9	19.5192	0.670807	14.1108

Reduced Basis – Efficiency in Black-Scholes Model

FEM-calculation with 801 basis functions: 35 seconds
Computation of the POD-basis: 39 seconds

σ	$C_A(0, 100)$	Delta	Vega
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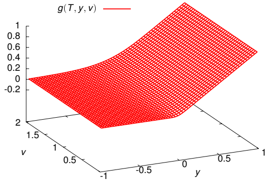
Reduced Basis – Efficiency in Black-Scholes Model

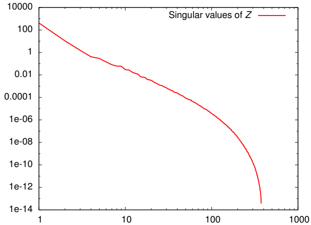
FEM-calculation with 801 basis functions: 35 seconds
Computation of the POD-basis: 39 seconds
POD-calculation with 15 basis functions: 1.2 seconds

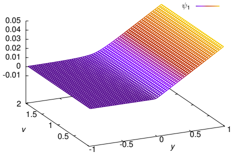
σ	$C_A(0,100)$	Delta	Vega	Price error	Delta error	Vega error
0.1	10.5968	0.989825	0.34534	$8.3006e-05$	$8.38586e-05$	0.0202638
0.188889	10.8198	0.926762	5.12669	0.000685423	0.000326805	0.0238353
0.277778	11.4769	0.845723	9.21924	0.000444494	0.000169424	0.0155903
0.366667	12.4005	0.787226	11.3620	0.000149486	0.000249543	0.0114599
0.455556	13.4670	0.748043	12.5556	0.000365294	0.000371058	0.000164472
0.544444	14.6142	0.720466	13.2412	0.000161619	0.000211238	0.00358165
0.633333	15.8085	0.700741	13.6519	0.000218763	0.000282165	0.00455001
0.722222	17.0313	0.686913	13.8967	0.000465986	0.000449245	$7.45315e-05$
0.811111	18.2704	0.677614	14.0334	$4.05614e-05$	$2.066e-05$	0.00913167
0.9	19.5182	0.671855	14.0983	0.00100108	0.00104796	0.0125393

Reduced Basis for the Heston Model

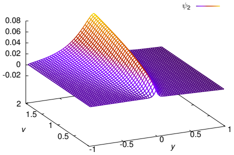
Example: Calculate $g(t, y, v)$ with $N_y = 61, N_v = 41, M = 625$



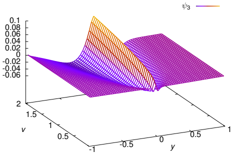




ψ_1



ψ_2



ψ_3

Reduced Basis – Efficiency in Heston Model

FEM-calculation with 81×61 basis functions: 123 seconds

θ_i	$C_A(0, 100, 0.022)$	Delta	Vega
1	11.421172	0.885409	4.287797
2	11.832293	0.850130	5.537867
3	12.015476	0.851747	4.963191
4	11.716659	0.852718	5.177694
5	10.913815	0.948827	3.548435
6	11.540681	0.863221	4.433649
7	11.465448	0.883098	6.339542
8	11.307237	0.888459	8.173694
9	11.147410	0.914950	5.961920
10	11.098593	0.900337	3.640245
11	10.985362	0.936262	5.988487
12	11.286456	0.904651	7.177647
13	10.858524	0.958111	1.823538
14	10.945517	0.941020	4.182806
15	11.559169	0.877870	5.840867

Reduced Basis – Efficiency in Heston Model

FEM-calculation with 81×61 basis functions: 123 seconds
Computation of the POD-basis: 132 seconds

θ_i	$C_A(0, 100, 0.022)$	Delta	Vega
1	11.421172	0.885409	4.287797
2	11.832293	0.850130	5.537867
3	12.015476	0.851747	4.963191
4	11.716659	0.852718	5.177694
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13	10.858524	0.958111	1.823538
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15	11.559169	0.877870	5.840867

Reduced Basis – Efficiency in Heston Model

FEM-calculation with 81 $\times$ 61 basis functions: 123 seconds
Computation of the POD-basis: 132 seconds
POD-calculation with 42 basis functions: 1.7 seconds

θ_i	$C_A(0, 100, 0.022)$	Delta	Vega	Price error	Delta error	Vega error
1	11.402936	0.885104	4.485977	0.018236	0.000305	0.198180
2	11.842398	0.851246	5.364510	0.010105	0.001115	0.173357
3	12.022748	0.853884	4.436634	0.007271	0.002136	0.526557
4	11.715899	0.853218	5.118843	0.000761	0.000500	0.058851
5	10.910492	0.947674	3.534090	0.003323	0.001153	0.014345
6	11.539249	0.862584	4.491026	0.001432	0.000637	0.057377
7	11.457013	0.884826	6.363838	0.008435	0.001728	0.024296
8	11.312174	0.887855	8.136989	0.004937	0.000604	0.036705
9	11.150456	0.916820	5.748191	0.003046	0.001870	0.213729
10	11.090177	0.903810	3.242317	0.008416	0.003473	0.397928
11	10.986913	0.936491	6.183714	0.001551	0.000229	0.195227
12	11.296781	0.907058	6.747697	0.010326	0.002407	0.429950
13	10.849729	0.959915	1.469688	0.008794	0.001804	0.353850
14	10.949302	0.939686	4.321778	0.003785	0.001334	0.138972
15	11.547142	0.879573	5.907704	0.012027	0.001704	0.066837

Approach

Scenario Generation

1. Choose a “true model” as well as its $\mathbb{P}$-and $\mathbb{Q}$ parameters.
2. Generate $N = 50000$ trajectories each with 63 days under $\mathbb{P}$ with daily observation of S_t and calculate daily $\mathbb{Q}$-prices of $C^{E,1}, \ldots, C^{E,15}$.

Approach

Scenario Generation

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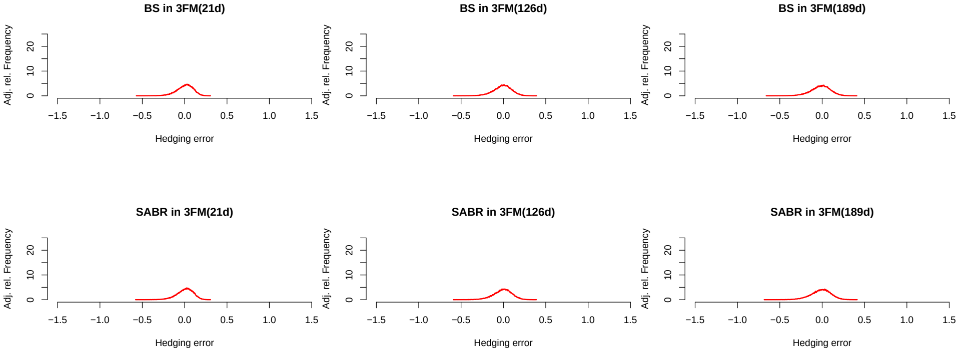
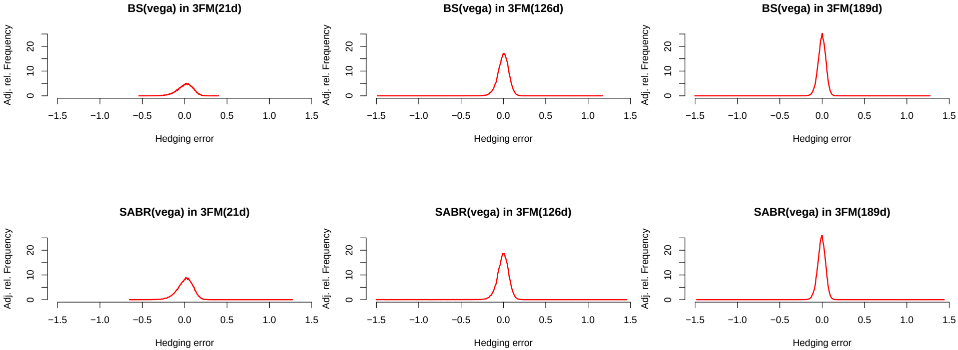
Hedge Calculation and Evaluation

1. Choose the maturity of C^A as $\{21, 126, 189\}$ days.
2. Choose the hedge model.
3. For each trajectory the hedge model gets calibrated to S_t and (a subset of) $C^{E,1}, \ldots, C^{E,15}$ on a daily basis.
4. Calculate the hedge portfolio.
5. At the end of the path the hedging error is evaluated.
6. Build up the hedging error distribution.

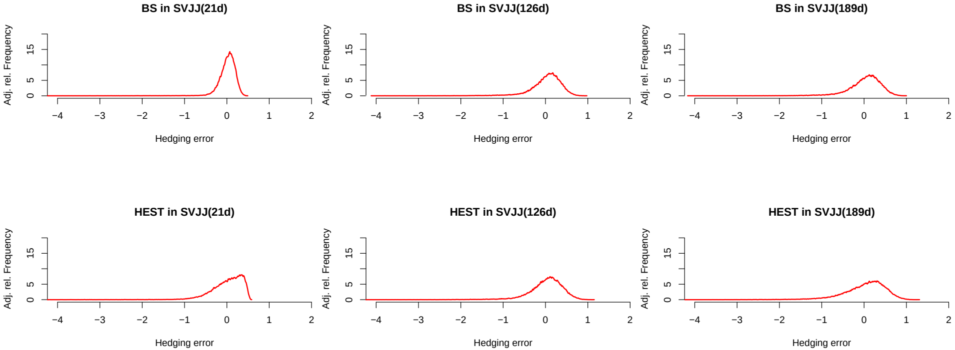
3FM, Local Calibration, Delta-Hedge



3FM, Local Calibration, Delta- and Vega-Hedge



SVJJ, Local Calibration, Delta-Hedge



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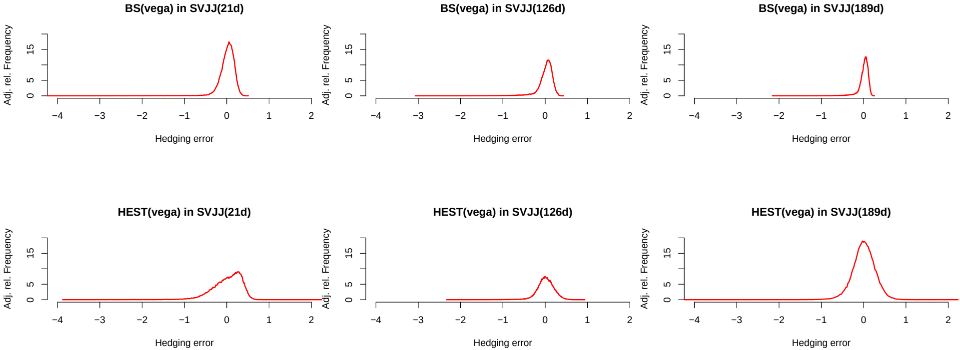
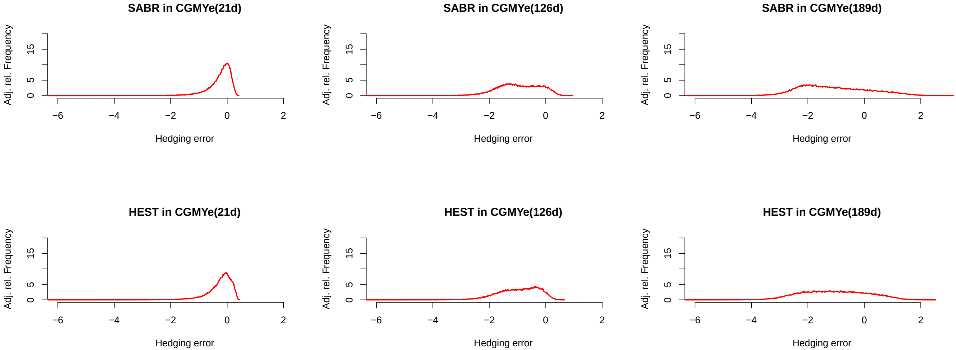
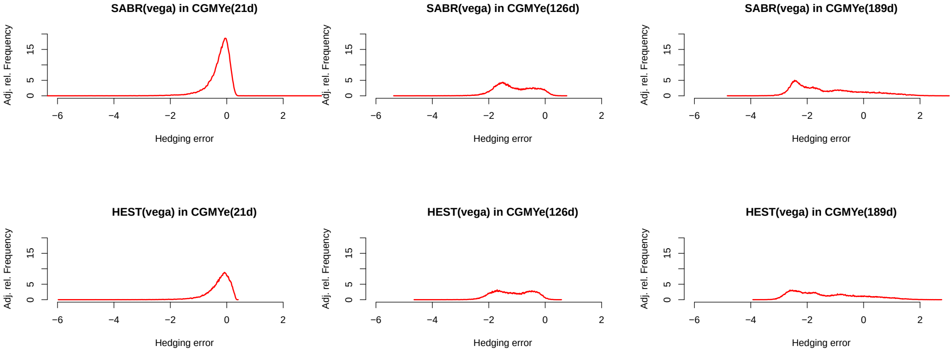


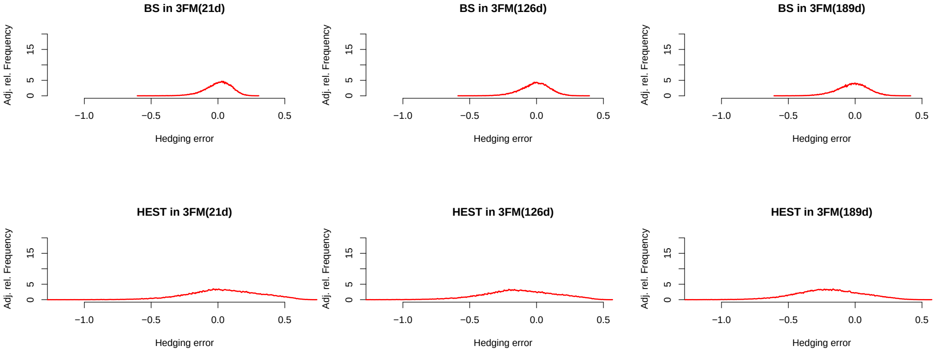
CGMYe, Local Calibration, Delta-Hedge



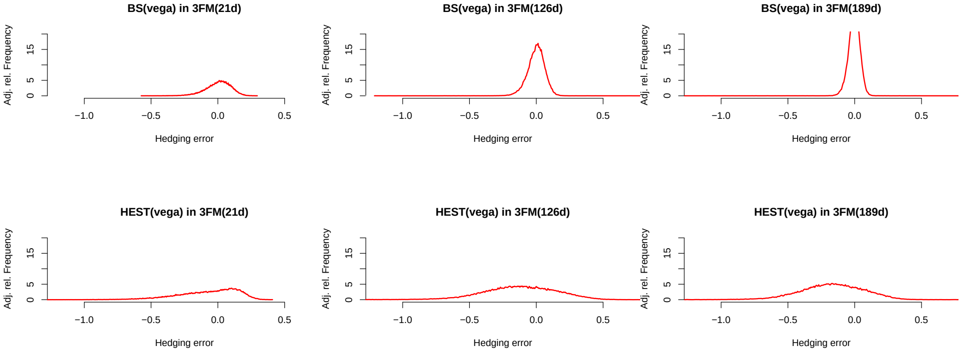
CGMYe, Local Calibration, Delta- and Vega-Hedge



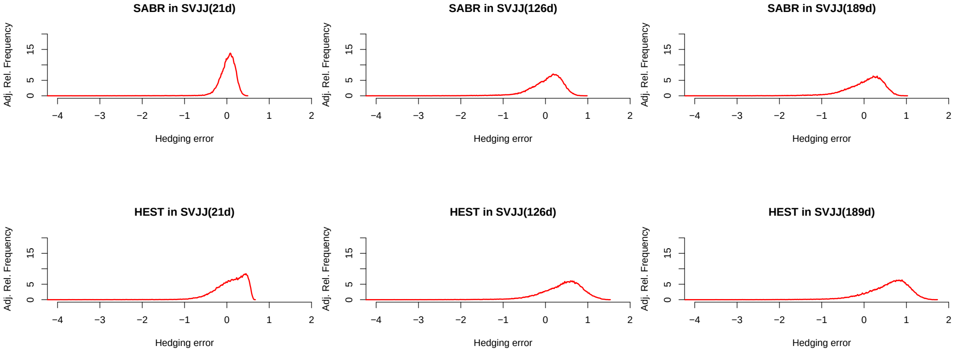
3FM, Global Calibration, Delta-Hedge



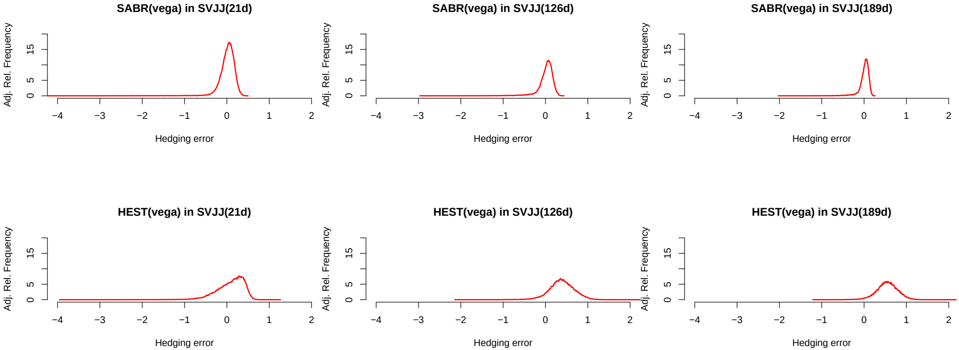
3FM, Global Calibration, Delta- and Vega-Hedge



SVJJ, Global Calibration, Delta-Hedge



SVJJ, Global Calibration, Delta- and Vega-Hedge



Wrap up

- Hedging under model uncertainty
- POD for parameter dependent (parabolic) PDEs
- Analysis of the hedging error distribution
- Simple models (BS) seem to be preferable in unknown markets

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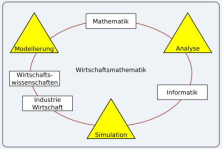
Contact



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Thank you for your attention