Double Mean-Reversion in FX

Philippe Balland

Quan Minh Tran

Gabriela Hodinic

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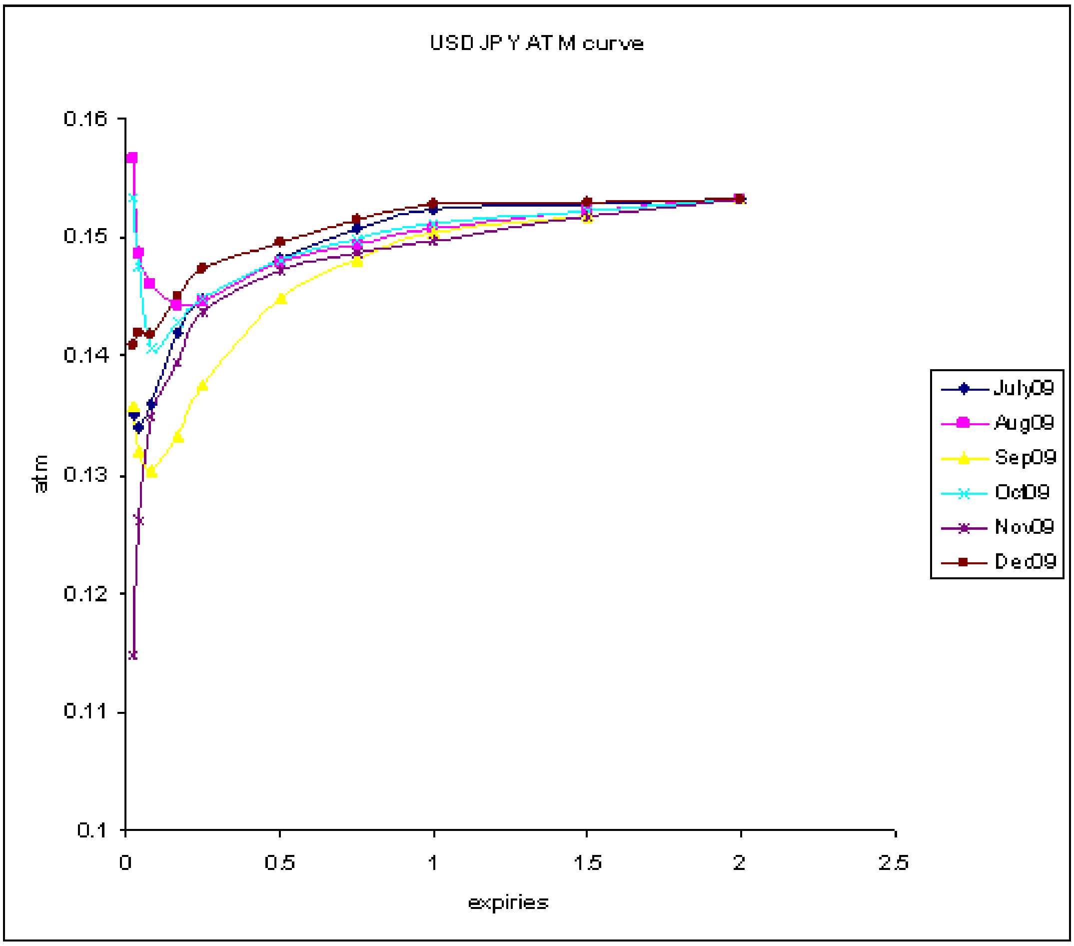
Market Observations

Three key observations:

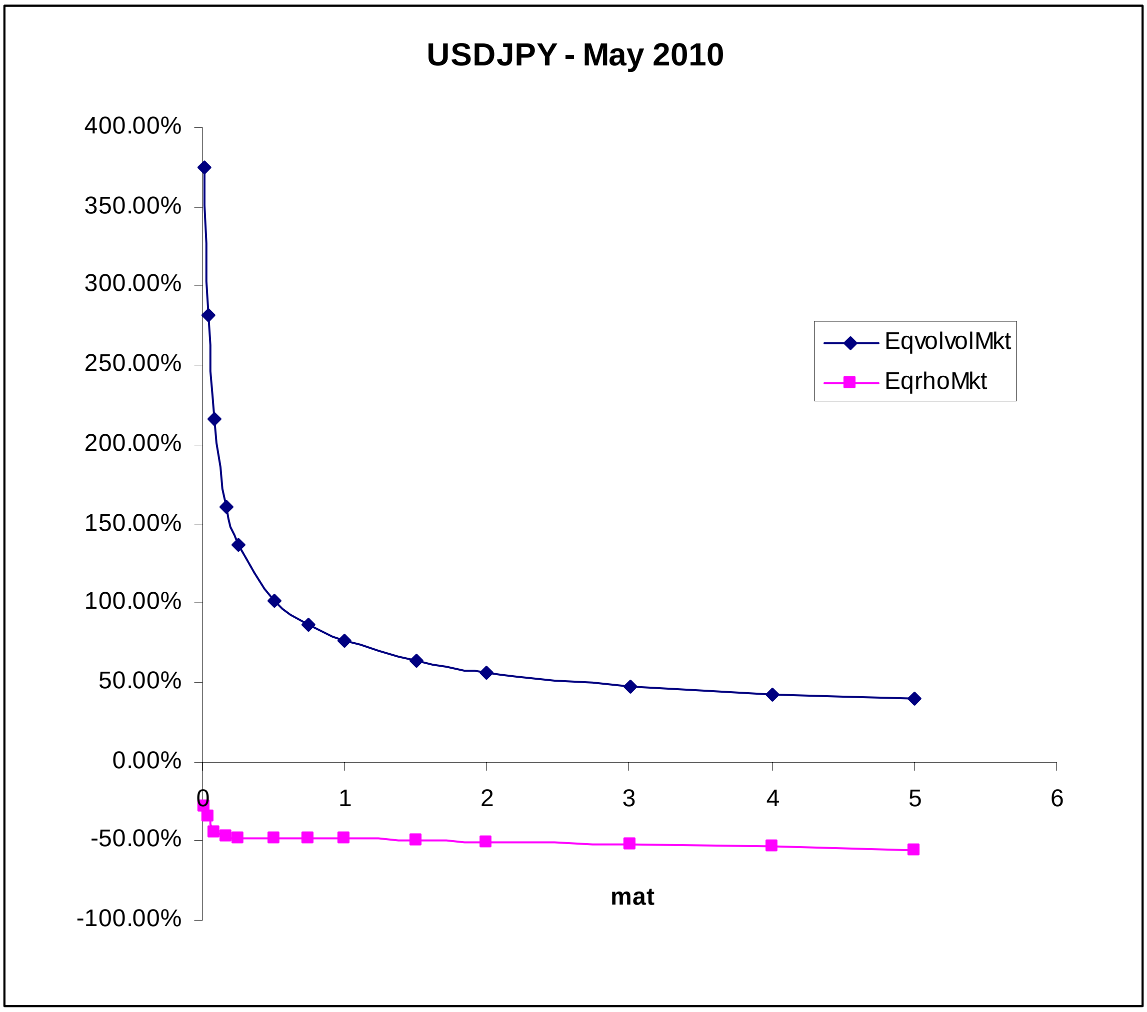
(1) Short-dated ATM expiring within a month are:

- highly volatile
- weakly correlated to ATM expiring beyond one year (when considering monthly changes).



(2) The ATM curve has often a local minimum within the first three months.

(3) Strangle margins are persistent at the short-end of the curve and the implied volatility-of-volatility is therefore large at the short-end.



Eqvolvol, *Eqrho* for expiry T are respectively the SABR volatility-of-volatility γ_{eq}^T and the SABR correlation ρ_{eq}^T:

$$
\begin{aligned}
&S_t = F(t)X_t \\
&dX_t / X_t = \sigma_0^T \exp(\gamma_{eq}^T W_t^\sigma - \tfrac{1}{2}(\gamma_{eq}^T)^2 t) dW_t^{fx} \\
&< dW_t^\sigma, dW_t^{fx} >= \rho_{eq}^T dt
\end{aligned}
$$

where S is the spot process, F is the forward, and σ_0^T is the initial volatility for expiry T.

We calculate γ_{eq}^T and ρ_{eq}^T using the local-time approximation as this is more accurate than the expansion formula in the vicinity of the forward:

$$
\begin{aligned}
&E[(X_T - k)^+] = (X_0 - k)^+ + \tfrac{k^2}{2}\int_0^T E[\sigma_t^2 \delta(X_t - k)] dt \\
&E[\sigma_t^2 \delta(X_t - k)] \approx \tfrac{A}{\sqrt{t}} e^{\kappa(k)t - \frac{1}{2t} I(k)^2}
\end{aligned}
$$

See E. Benhamou, O. Croissant (2007).

Single Mean-Reversion

We consider for each expiry T the following one-factor mean-reverting dynamic:

$$dX_t / X_t = \sigma_0^T \sqrt{\upsilon_t} dW_t^{fx}$$

$$\upsilon_t = \exp(2Z_t^\sigma - 2\operatorname{var} Z_t^\sigma)$$

$$Z_t^\sigma = \int_0^t \gamma\, e^{\lambda(u-t)} dW_u^\sigma$$

$$< dW_t^\sigma, dW_t^{fx} >= \rho dt$$

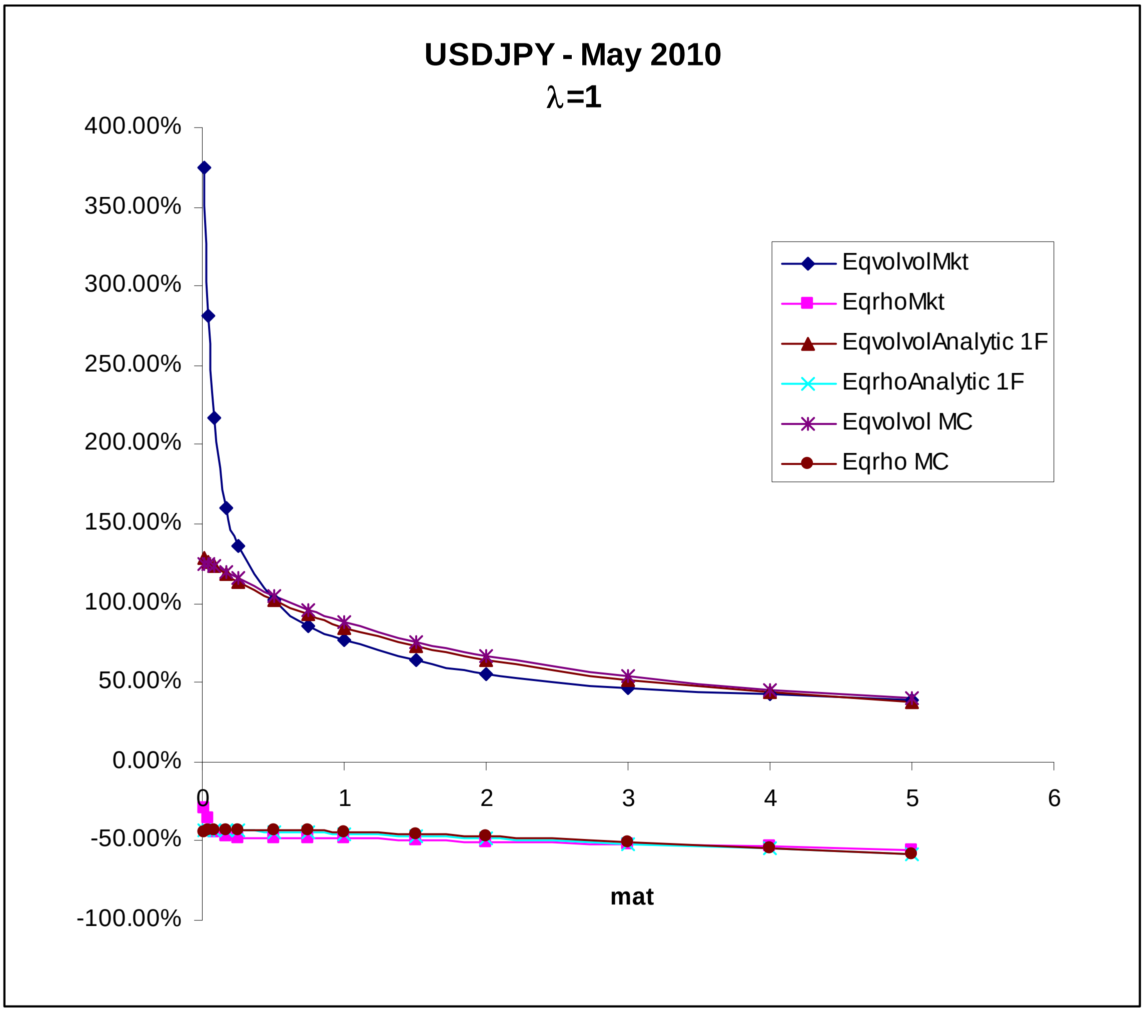
The volatility-of-volatility γ, correlation ρ and mean-reversion λ are calibrated to the entire smile-surface by moment matching:

$$K_{1F}[\gamma, \lambda](T) = K_{1F}[\gamma_{eq}^T, 0](T)$$

$$K_{1F}[\gamma, \lambda](T) = \frac{E[(\int_0^T \sigma_u^2 du)^2]}{E[(\int_0^T \sigma_u^2 du)]^2}$$

$$\rho = \rho_{eq}^T$$

As illustrated in the example below, the calibration loses accuracy at the short-end. This could be prevented by using a time-dependent volatility-of-volatility, but at the cost of losing time-homogeneity.



Double Mean-Reversion

We could improve the calibration's accuracy at the short-end and maintain time-homogeneity by using a two-factor stochastic volatility model:

$$
\begin{aligned}
& dX_t / X_t = \sigma_0^T \sqrt{\upsilon_t} dW_t^{fx} \\
& \upsilon_t = \exp(2Z_t^\sigma - 2\operatorname{var} Z_t^\sigma) \\
& dZ_t^\sigma = \lambda_{ld}(Z_t^{sd} - Z_t^\sigma)dt + \gamma_{ld} dW_t^{ld} \\
& dZ_t^{sd} = \lambda_{sd}(Z_\infty - Z_t^{sd})dt + \gamma_{sd} dW_t^{sd}
\end{aligned}
$$

By direct integration, we obtain:

$$
\begin{aligned}
Z_t^\sigma &= Z_0^\sigma e^{-\lambda_{sd} t} + Z_\infty (1 - e^{-\lambda_{sd} t}) + (Z_0^\sigma - Z_\infty) \tfrac{\lambda_{sd} - \lambda_{ld}}{\lambda_{sd}} (e^{-\lambda_{ld} t} - e^{-\lambda_{sd} t}) \\
&+ \tfrac{\lambda_{sd}}{\lambda_{sd} - \lambda_{ld}} \int_0^t \gamma_{ld} e^{\lambda_{ld}(u-t)} (1 - e^{(\lambda_{sd} - \lambda_{ld})(u-t)}) dW_u^{ld} \\
&+ \int_0^t \gamma_{sd} e^{\lambda_{sd}(u-t)} dW_u^{sd} \\
&= E[Z_t^\sigma] + Z_t^{ld} + Z_t^{sd}
\end{aligned}
$$

Typically, the dynamic is based on two non-overlapping time-scales:

$$
\tau_{sd} = 1/\lambda_{sd} \propto 1 week, \quad \tau_{ld} = 1/\lambda_{ld} \propto 1 year.
$$

Hence, the dynamic is similar to the two-factor volatility model considered in *Balland* (2006) and *Bergomi* (2008, 2005):

$$
\begin{aligned}
Z_t^{ld} &= \tfrac{\lambda_{sd}}{\lambda_{sd}-\lambda_{ld}} \int_0^t \gamma_{ld} e^{\lambda_{ld}(u-t)} (1 - e^{(\lambda_{sd}-\lambda_{ld})(u-t)}) dW_u^{ld} \\
&\approx \int_0^t \gamma_{ld} e^{\lambda_{ld}(u-t)} dW_u^{ld}
\end{aligned}
$$

$$
Z_t^{sd} = \int_0^t \gamma_{sd} e^{\lambda_{sd}(u-t)} dW_u^{sd}
$$

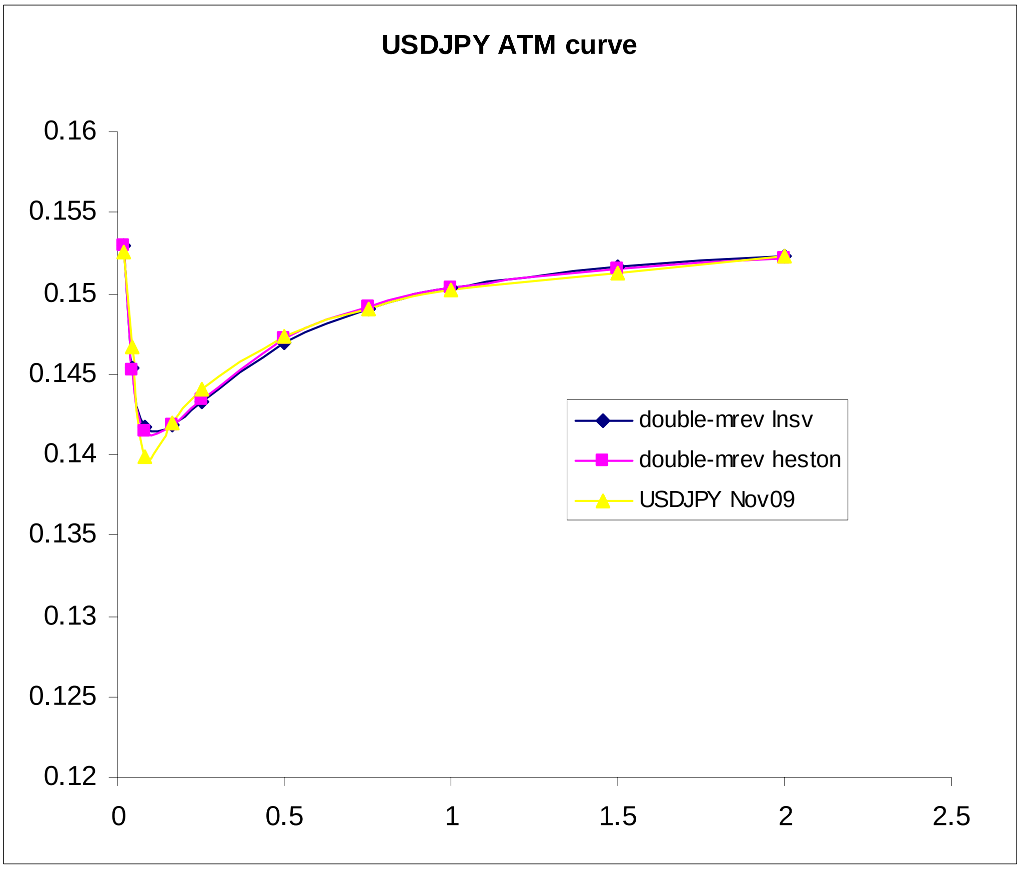
ATM Minimum

We can approximate short-dated ATM levels as follows:

$$ATM_T \approx \sqrt{\tfrac{1}{T}\int_0^T E[\sigma_u^2]du}$$

$$\tfrac{1}{T}\int_0^T E[\sigma_u^2]du = \tfrac{1}{T}\int_0^T \exp(2E[Z_u^\sigma])du$$

The ATM curves generated by the double mean-reversion model admit typically local minima.



Smile Calibration

We calibrate the parameters $\gamma_{sd}, \gamma_{ld}, \lambda_{sd}, \lambda_{ld}$ to the smile surface by moment matching:

$$K_{2F}[\gamma_{sd}, \gamma_{ld}, \lambda_{sd}, \lambda_{ld}](T) = K_{1F}[\gamma_{eq}^{T}, 0](T)$$

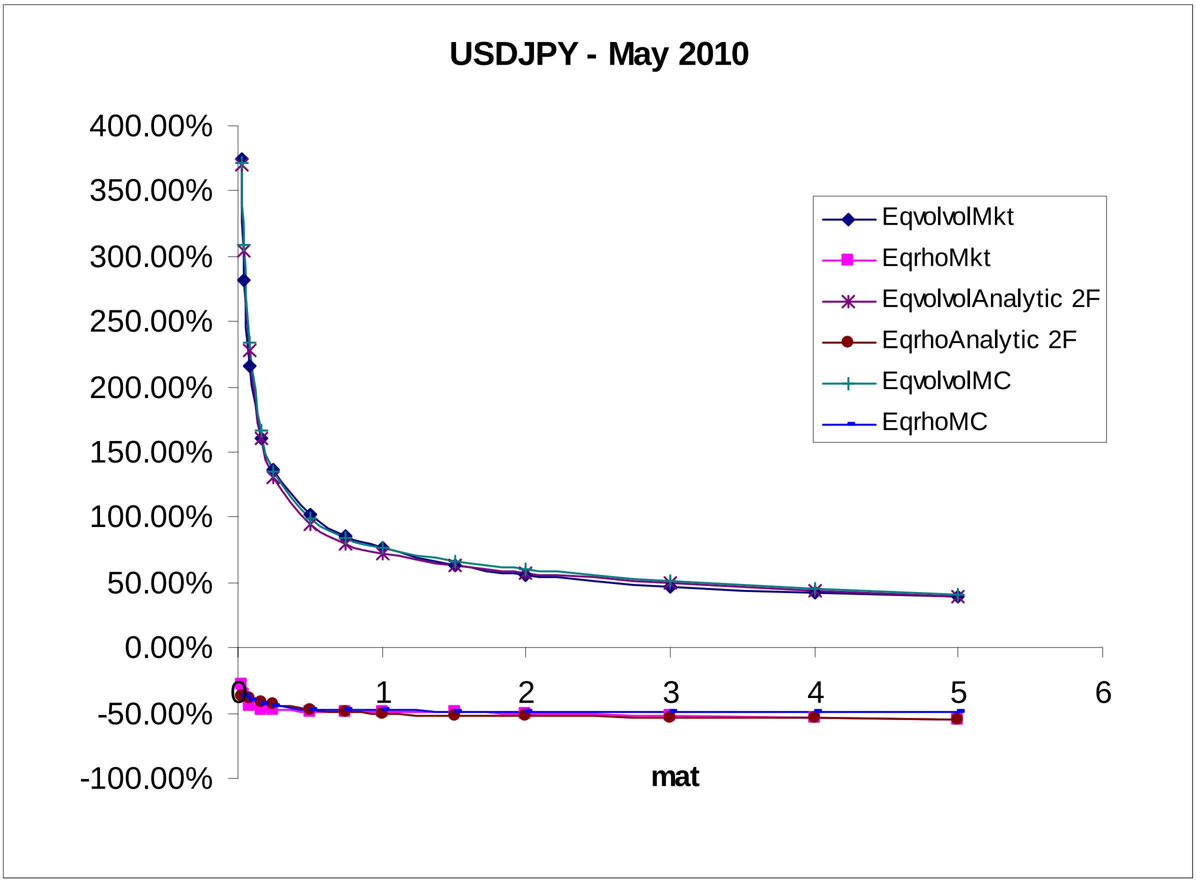
$$K_{2F}[\gamma_{sd}, \gamma_{ld}, \lambda_{sd}, \lambda_{ld}](T) = \frac{E[(\int_0^T \sigma_u^2 du)^2]}{E[(\int_0^T \sigma_u^2 du)]^2}$$

$$\frac{\rho_{sd,fx}\gamma_{eq,sd}^{T} + \rho_{ld,fx}\gamma_{eq,ld}^{T}}{\sqrt{(\gamma_{eq,sd}^{T})^2 + (\gamma_{eq,ld}^{T})^2 + 2\rho_{sd,ld}\gamma_{eq,sd}^{T}\gamma_{eq,ld}^{T}}} = \rho_{eq}^{T}$$

where $\gamma_{eq,sd}^{T}$ and $\gamma_{eq,ld}^{T}$ are obtained by solving the following equations:

$$K_{1F}[\gamma_{eq,sd}^{T}, 0](T) = K_{1F}[\gamma_{sd}, \lambda_{sd}](T)$$

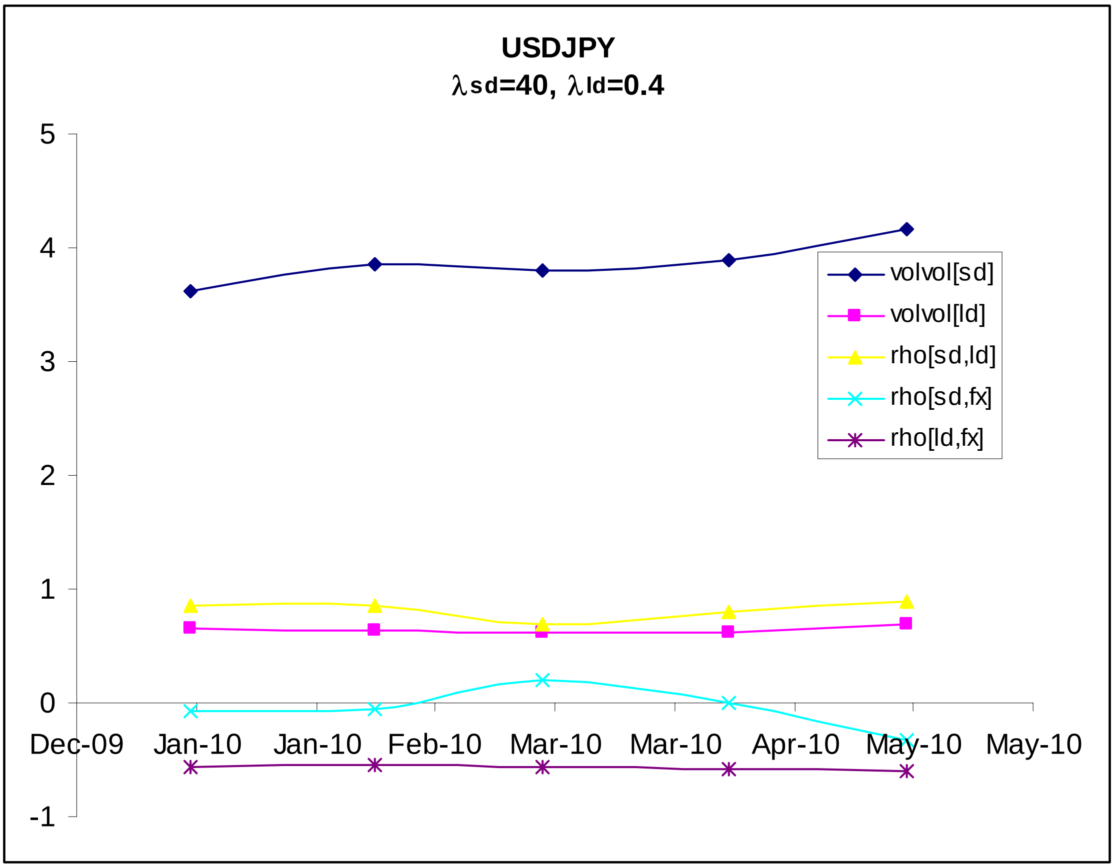
$$K_{1F}[\gamma_{eq,ld}^{T}, 0](T) = K_{1F}[\gamma_{ld}, \lambda_{ld}](T)$$



The moment-matching technique used to compute effective volatility-of-volatilities and correlations is accurate when compared to the Monte-Carlo method.

We observe that the fit to market data is substantially improved by using double mean-reversion.

As illustrated in the example below, the model parameters $\lambda_{sd}, \lambda_{ld}$, γ_{sd}, γ_{ld}, $\rho_{sd,ld}$ and $\rho_{ld,fx}$ appear relatively stable over time.



The correlation parameter $\rho_{sd,fx}$ appears less stable. This is expected since the model does not include any local volatility.

Two-Factor SLV Model

We control the joint evolution between risk-reversal and spot by including a local volatility component $\sigma(t, \ln X_t)$ in the dynamic:

$$S_t = X_t F_{0t}$$

$$dX_t / X_t = \sigma(t, \ln X_t)\sqrt{\upsilon_t}\, dW_t^{fx}$$

$$\upsilon_t = \exp(2Z_t^{\sigma} - 2\operatorname{var} Z_t^{\sigma})$$

$$Z_t^{\sigma} = Z_t^{ld} + Z_t^{sd}$$

$$\begin{aligned} Z_t^{ld} &= \int_0^t m_u^{ld} \rho_{ld,fx} \gamma_{ld}\, e^{\lambda_{ld}(u-t)} dW_u^{fx} \\ &+ \int_0^t m_u^{ld\perp} (1-\rho_{ld,fx}^2)^{1/2} \gamma_{ld}\, e^{\lambda_{ld}(u-t)} dW_u^{fx\perp} \end{aligned}$$

$$\begin{aligned} Z_t^{sd} &= \int_0^t m_u^{sd} \rho_{sd,fx} \gamma_{sd}\, e^{\lambda_{sd}(u-t)} dW_u^{fx} \\ &+ \int_0^t m_u^{sd\perp} \alpha_{sd} \gamma_{sd}\, e^{\lambda_{sd}(u-t)} dW_u^{fx\perp} \\ &+ \int_0^t m_u^{sd\perp} (1-\rho_{sd,fx}^2 - \alpha_{sd}^2)^{1/2} \gamma_{sd}\, e^{\lambda_{sd}(u-t)} dW_u^{fx\perp\perp} \end{aligned}$$

$$\alpha_{sd} = \frac{\rho_{sd,ld} - \rho_{sd,fx}\rho_{ld,fx}}{(1-\rho_{ld,fx}^2)^{1/2}}$$

where $m^{sd}, m^{sd\perp}, m^{ld}, m^{ld\perp}$ are the mixing-weight parameters.

As these parameters vary between zero and one while maintaining fixed the target smile, the dynamic varies from local to stochastic volatility dynamic.

The parameters m^{sd}, m^{ld} control the amount of volatility-of-volatility parallel to the spot motion. As they increase from zero to one while the target smile is fixed, the slope of the local volatility decreases to compensate for the increase in volatility-of-volatility parallel to spot. Despite these parameters affecting the backbone of the dynamic, they have in fact little effect on the valuation of exotics.

An asymptotic calculation shows that we have for all mixing weights:

$$\frac{\Delta ATM}{\Delta \ln F} = 2(\partial_{\ln K/F} \Sigma_0)(\ln F, 0)$$

$$\Sigma_T(\ln F, \ln K/F) := \text{implied-volatility}$$

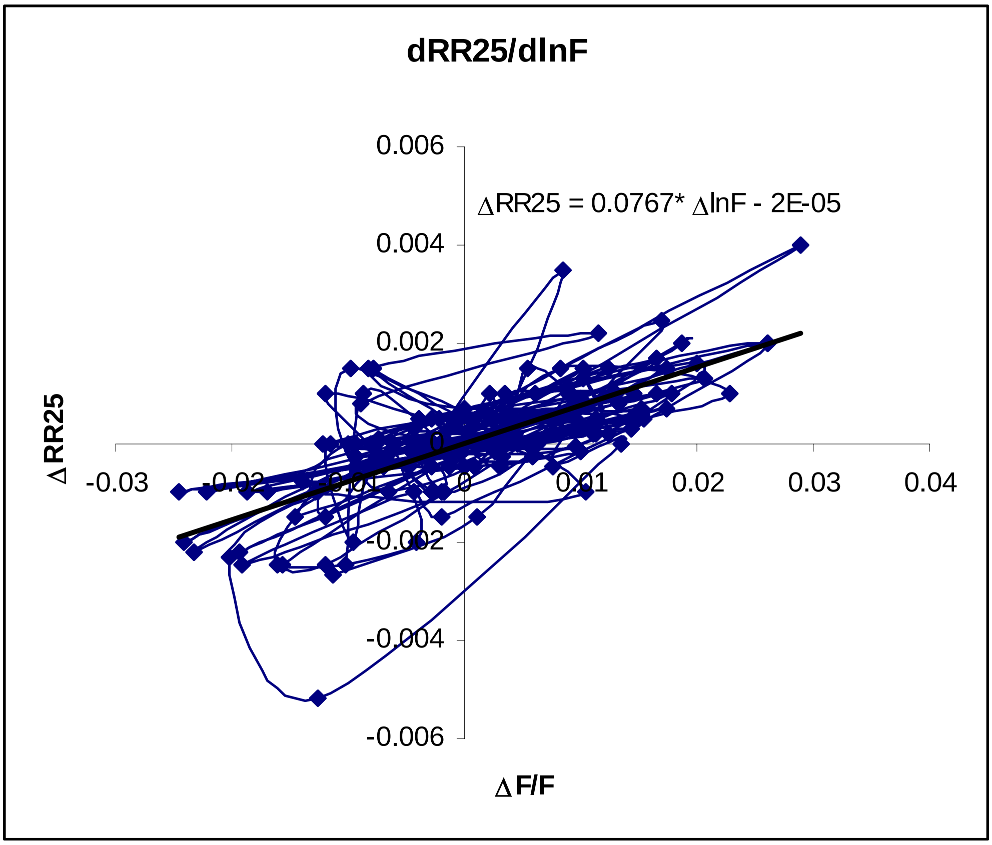
The ATM-speed coefficient $\frac{\Delta ATM}{\Delta \ln F}$ is to be understood in the sense of Malliavin derivative:

$$\frac{dATM_t}{ATM_t} = \frac{\Delta ATM}{\Delta \ln F} dW_t^{fx} + (\cdots) dW_t^{fx\perp}$$

The parameters $m^{sd\perp}, m^{ld\perp}$ control the amount of volatility-of-volatility orthogonal to the spot motion. As they increase from zero to one while the target smile is fixed, the convexity of the local volatility decreases to compensate for the increase in volatility-of-volatility.
Hence, the mixing-weights $m^{ld\perp}, m^{sd\perp}$ control the convexity of the local volatility and thus control the joint evolution of risk-reversal (slope of smile) and spot.
They are therefore critical to the valuation of Barrier and DNT products as these parameters affect directly the expected slope of the smile prevailing when spot hits the barrier level.

An asymptotic calculation shows that the RR25 speed $\frac{\Delta RR25}{\Delta \ln F}$ depend on the level of mixing weights.

As illustrated in the example below, the spot and risk-reversal are strongly correlated.



The mixing weight parameters $m^{ld\perp}, m^{sd\perp}$ can be set to match historical RR25 speeds or DNT prices.

Simplification

Typical short-dated products do not depend strongly on the mixing-weights m^{sd} and m^{ld} as these parameters control whether the skew implied by the dynamic originates from local or stochastic volatility.
Consequently, we can simplify the dynamic by assuming m^{sd} and m^{ld} to be zero:

$$
\begin{aligned}
&S_t = X_t F_{0t} \\
&dX_t / X_t = \sigma(t, \ln X_t)\sqrt{\upsilon_t}\, dW_t^{fx} \\
&\upsilon_t = \exp(2Z_t^{\sigma} - 2\operatorname{var} Z_t^{\sigma}) \\
&Z_t^{\sigma} = \int_0^t [m_u^{ld\perp}(1-\rho_{ld,fx}^2)^{1/2}\gamma_{ld} + m_u^{sd\perp}\alpha_{sd}\gamma_{sd}\, e^{(\lambda_{sd}-\lambda_{ld})(u-t)}]\, e^{\lambda_{ld}(u-t)} dW_u^{fx\perp} \\
&\qquad + \int_0^t m_u^{sd\perp}(1-\rho_{sd,fx}^2 - \alpha_{sd}^2)^{1/2}\gamma_{sd}\, e^{\lambda_{sd}(u-t)} dW_u^{fx\perp\perp} \\
&\alpha_{sd} = \frac{\rho_{sd,ld} - \rho_{sd,fx}\rho_{ld,fx}}{(1-\rho_{ld,fx}^2)^{1/2}}
\end{aligned}
$$

Using our moment matching technique, we can approximate the dynamic of the volatility driver using either a one-factor or a two-factor dynamic:

(i) $$Z_t^\sigma = \int_0^t \gamma_u^\perp \, e^{\lambda_{ld}(u-t)} dW_u^{fx\perp} + \int_0^t \gamma_u^{\perp\perp} \, e^{\lambda_{sd}(u-t)} dW_u^{fx\perp\perp}$$

(ii) $$Z_t^\sigma = \int_0^t \gamma_u^\perp \, e^{\lambda_{ld}(u-t)} dW_u^{fx\perp}$$

The orthogonalisation allows fast backward and forward inductions. In particular, we can approximate the volatility drivers using Markov chains:

(i) $$Z_t^\sigma = \Sigma_t^\perp \xi_t^\perp + \Sigma_t^{\perp\perp} \xi_t^{\perp\perp}$$

(ii) $$Z_t^\sigma = \Sigma_t^\perp \xi_t^\perp$$

where $\xi_t^\perp, \xi_t^{\perp\perp}$ are independent $N(0,1)$-processes characterized by their auto-correlation functions.

The version (ii) is sufficient for first generation exotic products. We can calibrate the local-volatility to the smile assuming (ii) in particular.

Calibration

Parametric Local Volatility

We parameterize the local volatility:

$$\sigma(\ln X_t) = \sigma_0(t) \\ \times \sigma_{skew}(\ln X_t) \\ \times \sigma_{convex}(\ln X_t) \\ \times \sigma_{killing}(\ln X_t)$$

We choose the local volatility skew using a ratio of CEV. This ensures that the skew component has a CEV-like shape near the forward while being bounded:

$$\sigma_{skew}(\ln X_t) = \frac{(X_t)^{\beta-1}+a}{(X_t)^{\beta-1}+b} \\ = \frac{1+q}{2} + \frac{1-q}{2}\tanh\left(\frac{1+q}{1-q}(\beta-1)\ln X_t\right)$$

$$q < \sigma_{skew}(\ln X_t) < 1$$

The local volatility skew component has a functional form similar to that suggested by *Brown* and *Randall* (2003):

$$\sigma(\ln X_t) = \sigma_{atm} + \sigma_{skew} \times \tanh(a_{skew} \ln X_t / X_{skew}) + \sigma_{smile} \times (1 - \frac{1}{\cosh(a_{smile} \ln X_t / X_{smile})})$$

Note however that the BR functional form is additive while our parameterisation is in fact multiplicative.

The convex local volatility shares the same short-dated asymptotic as SABR in order to minimize changes in the smile when the mixing-weight parameters vary:

$$\sigma_{convex}(\ln X_t) = \sqrt{1 + 2a \ln X_t + b^2 (\ln X_t)^2}$$

The killing component ensures finite moments by exponentially decreasing the spot volatility outside the boundaries $X_{low}(t)$ and $X_{up}(t)$:

$$\sigma_{killing}(\ln X_t) = \exp(-k(\ln \frac{X_t}{X_{up}(t)})^+ - k(\ln \frac{X_{low}(t)}{X_t})^+)$$

Finally, we calibrate the function $\sigma_0(t)$ to the prevailing ATM curve by forward induction.

Parametric Smile

We can parameterize the smile surface using asymptotic expansions of the previous diffusion $(\gamma_{sd} = 0, \lambda_{ld} = 0)$ which is a direct extension of SABR for FX:

$$dX_t \,/\, X_t = \sigma_t \times \sigma_{skew}(\ln X_t) \times \sigma_{convex}(\ln X_t)\, dW_t$$
$$\sigma_t = \sigma_0^T \exp(\gamma^T W_t^\sigma - \tfrac{1}{2}(\gamma^T)^2 t)$$

In this case, the local volatility is obtained by forward induction using the equation:

$$\sigma(t, \ln X_t) = \sigma_D(t, \ln X_t) \,/\, E[e^{2Z_t^\sigma - 2\operatorname{var} Z_t^\sigma} \mid \ln X_t]^{1/2}$$

where $\sigma_D(t, \ln X_t)$ is the Dupire local volatility obtained from the parameterized smile.