DERIVATIVE TIME SCALE PERTURBATIONS

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OUTLINE AND OBJECTIVES

- ↪ Stochastic volatility modeling.
- ↪ Illustration equities.
- ↪ On model and parameters interpretation.
- ↪ Role of skew in credit markets.

Aspects and objectives:

◇ "Hidden" Volatility/Parameter Time Scales and parameter heterogeneity are important; leverage and clustering effects.

◇ Efficient and simple description of Stochastic Volatility effects using Perturbation Methods, under separartion of time scales.

◇ Parsimonious "Effective Parameteric" representation for derivative Linkage and insight captured by perturbations.

Volatility Time Scales

- **Rescale the time of a diffusion process Y_t^1:**

$$Y_t^\alpha = Y_{\alpha t}^1$$

α **large** $\longrightarrow$ **"speeding up" the process** Y_t^1
α **small** $\longrightarrow$ **"slowing down" the process** Y_t^1
$1/\alpha$ **is the** *characteristic time scale* **of the process** Y_t^α**.**

- **On effective volatility, fast scale case:**

$$\overline{\sigma^2}(0,T) \equiv \frac{1}{T}\int_0^T f^2(Y_t^\alpha)dt \ = \ \frac{1}{T'}\int_0^{T'} f^2(Y_s^1)ds\,, \qquad \mathbf{T}' \equiv \alpha\mathbf{T} \to +\infty$$

Assuming that Y^1 is ergodic with invariant distribution Φ_Y then:

$$\lim_{T'\to+\infty} \frac{1}{T'}\int_0^{T'} f^2(Y_s^1)ds = \int f^2(y)\Phi(dy) \equiv \langle f^2\rangle_{\Phi_Y}.$$

Effective volatility: $\bar{\sigma}^2 \equiv \langle f^2\rangle_{\Phi_Y}$.

Averaging and the Correction; Fast Scale

$$dY_t^1 = c(Y_t^1)dt + g(Y_t^1)dW_t\,, \quad Y_0^1 = y$$

$$dY_t^\alpha \stackrel{\mathcal{D}}{=} \alpha c(Y_t^\alpha)dt + \sqrt{\alpha}\, g(Y_t^\alpha)dW_t\,, \quad Y_0^\alpha = y$$

Fast "oscillating" integral:

$$\mathcal{H}(\alpha;T) \equiv \frac{1}{T}\int_0^T \left(f^2(Y_s^\alpha) - \bar{\sigma}^2\right) ds = \frac{1}{T}\int_0^T \mathcal{L}_{Y1}\phi(Y_s^\alpha)\, ds,$$

Poisson equation: $\mathcal{L}_{Y1}\phi(y) = f^2(y) - \bar{\sigma}^2$

Itô: $d\phi(Y_s^\alpha) = \alpha\mathcal{L}_{Y1}\phi(Y_s^\alpha)\, ds + \sqrt{\alpha}\,\phi'(Y_s^\alpha)g(Y_s^\alpha)dW_s$

$$\mathcal{H}(\alpha;T) = \frac{1}{\alpha T}\int_0^T d\phi(Y_s^\alpha) - \frac{1}{\sqrt{\alpha}T}\int_0^T \phi'(Y_s^\alpha)g(Y_s^\alpha)dW_s$$

$$= -\frac{1}{\sqrt{\alpha}}\left(\frac{1}{T}\int_0^T \phi'(Y_s^\alpha)g(Y_s^\alpha)dW_s\right) + \mathcal{O}\left(\frac{1}{\alpha}\right)$$

Correction :

(i) Correlation with the BM driving the underlying gives correction at the order $\sqrt{\alpha}$. (ii) Risk neutral Market price of volatility risk gives correction at same order.

Slowing Down the Time

Assuming f and g smooth,

$$\overline{\sigma^2}(0,T) = \frac{1}{T}\int_0^T f^2(Y_t^\alpha)dt \longrightarrow f^2(y) \quad \textbf{as} \quad \alpha \to \mathbf{0}$$

$$\tilde{\mathcal{H}}(\alpha;T,y) = \frac{1}{T}\int_0^T \left(f^2(Y_s^\alpha) - f^2(y)\right) ds$$

$$= \frac{1}{T}\int_0^T \left[f^2\left(y + \alpha\int_0^t c(Y_s^\alpha)ds + \sqrt{\alpha}\int_0^t g(Y_s^\alpha)dW_s\right) - f^2(y)\right] dt$$

$$= 2\sqrt{\alpha}\, f(y)f'(y)g(y)\left(\frac{1}{T}\int_0^T W_t dt\right) + \mathcal{O}(\alpha)$$

Correction : (i) Correlation with the BM driving the underlying gives correction at the order $\sqrt{\alpha}$ (ii) Risk neutral Market price of volatility risk gives correction at same order.

Multiscale Stochastic Volatility Models

The volatility is a function of two factors:

$$\sigma_t = f(Y_t, Z_t)$$

- Y_t **fast mean-reverting (ergodic):**

$$dY_t = \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^{(y)}, \quad 0 < \varepsilon \ll 1$$

- Z_t **is slowly varying:**

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}\, g(Z_t)dW_t^{(z)}, \quad 0 < \delta \ll 1$$

(y, z) **will denote the initial point for** (Y, Z)
Local Effective Volatility:

$$\bar{\sigma}^2(z) \equiv \left\langle f^2(\cdot, z)\right\rangle_{\Phi_Y}$$

Fouque et al, SIAM MMS 03.

Fast Scale Price; Motivation

↪ **Consider** (zero short rate):

$$dX_t = f(Y_t)X_t dW_t^{(x)}\,; \qquad dY_t = \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^{(y)}\,,$$

$$d\left\langle W^{(x)}, W^{(y)}\right\rangle = \rho_y\,.$$

- **The price:** $P_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$.

- **The approximation:** $\tilde{P}_t = M_t + R_t$ **with**

(i) $\tilde{P}_t = \tilde{P}(t, X_t)$, $\tilde{P}_T = h(x)$, **(ii)** M_t **martingale**, **(iii)** $R_t = \mathcal{O}(\varepsilon)$.

↪ **Then:**

$$\begin{aligned}\tilde{P}_t &= \mathbb{E}^\star[M_T + R_T \mid \mathcal{F}_t] + R_t - \mathbb{E}^\star[R_T \mid \mathcal{F}_t] \\ &= P_t + (R_t - \mathbb{E}^\star[R_T \mid \mathcal{F}_t]) = P_t + \mathcal{O}(\varepsilon)\,.\end{aligned}$$

Fast Scale Approximation

$$
\begin{aligned}
\tilde{P}_t - \tilde{P}_0 \quad &= \quad \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s\,ds + f(Y_s)X_s\partial_x\tilde{P}_s dW_s^{(x)} \\
&\quad + \frac{1}{2}\left(f^2(Y_s) - \bar{\sigma}^2\right) X_s^2\partial_x^2\tilde{P}_s\,ds \\
&\overset{\text{by Poisson}}{=} \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s\,ds + f(Y_s)X_s\partial_x\tilde{P}_s dW_s^{(x)} \\
&\quad + \frac{\varepsilon}{2}\left(d\phi - \frac{\beta(Y_s)\phi'(Y_s)}{\sqrt{\varepsilon}}dW^{(y)}\right) X_s^2\partial_x^2\tilde{P}_s \\
&\overset{\text{by parts}}{=} \int_0^t \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_s\,ds - \sqrt{\varepsilon}\,\langle\mathcal{V}(Y_s)\rangle_{\Phi_Y}\, ds(X_t\partial_x)(X_t^2\partial_x^2)P_t \\
&\quad + \sqrt{\varepsilon}\left(\mathcal{V}(Y_s) - \langle\mathcal{V}(Y_s)\rangle_{\Phi_Y}\right)(X_s\partial_x)(X_s^2\partial_x^2)P_s\,ds + ..dW_s^{(x)} + ..dW_s^{(y)} \\
&= \quad \text{martingale} + \mathcal{O}(\varepsilon)\,.
\end{aligned}
$$

$$
\begin{aligned}
d\left\langle \phi, x^2\partial_x^2 P\right\rangle_t &= \rho_y\beta(Y_t)f(Y_t)\phi'(Y_s)(X_t\partial_x)(X_t^2\partial_x^2)P_t dt \\
&\equiv \mathcal{V}(Y_t)(X_t\partial_x)(X_t^2\partial_x^2)P_t\,dt\,.
\end{aligned}
$$

Fast Scale; Defining Problem

$\hookrightarrow$ **Thus we want $\tilde{P} = \tilde{P}(t,x)$ so that:**

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_t - \sqrt{\varepsilon}\,\langle \mathcal{V}(Y_t)\rangle_{\Phi_Y}(x\partial_x)(x^2\partial_x^2)\tilde{P}_t \;=\; \mathcal{O}(\varepsilon)\,;\tilde{P}_T = h\,.$$

- **The case with market price of risk:**

$$dY_t \;=\; \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda(Y_t)\right)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\,dW_t^{(y)\star}\,,$$

gives $\varepsilon d\phi \mapsto \varepsilon d\phi + \sqrt{\varepsilon}\beta(Y_t)\Lambda(Y_t)\Phi'(Y_t)\,dt$**.**

$\hookrightarrow$ **Then we want**

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_t - \frac{\sqrt{\varepsilon}}{2}\left\langle\beta(Y_t)f(Y_t)\Phi'(Y_t)\right\rangle_{\Phi_Y}(x\partial_x)(x^2\partial_x^2)\tilde{P}_t$$
$$+\frac{\sqrt{\varepsilon}}{2}\left\langle\beta(Y_t)\Lambda(Y_t)\Phi'(Y_t)\right\rangle_Y(x^2\partial_x^2)P_t = \mathcal{O}(\varepsilon)\,;\tilde{P}_T = h\,.$$

Decomposition Fast Scale

- **Recall therefore:**

$$\mathcal{L}_{BS}(\bar\sigma)\tilde P_t + \sqrt{\varepsilon}\left(V_3(x\partial_x)(x^2\partial_x^2)\tilde P_t + V_2(x^2\partial_x^2)P_t\right) = \mathcal{O}(\varepsilon)\,,$$
$$\tilde P_T = h\,.$$

$\hookrightarrow$ **Decomposition** $\tilde P = P^{(0)} + \sqrt{\varepsilon}P^{(1)} + \mathcal{O}(\varepsilon)$.

$$\begin{aligned}\mathcal{L}_{BS}(\bar\sigma)(\,P^{(0)}) &= 0\,; & P^{(0)}|_T = h\,,\\ \mathcal{L}_{BS}(\bar\sigma)(\,P^{(1)}) &= (\sqrt{\varepsilon}\,\mathcal{A})P^{(0)}\,; & P^{(1)}|_T = 0\,.\end{aligned}$$

$$\mathcal{A} = -V_2(x^2\partial_x^2) - V_3(x\partial_x)(x^2\partial_x^2)\,,$$

$$V_2 = \frac{1}{2}\left\langle \beta\Lambda\phi'\right\rangle_{\Phi_Y}, V_3 = -\frac{\rho y}{2}\left\langle \beta f\phi'\right\rangle_{\Phi_Y}\,.$$

$\rightarrow$ **Can solve explicitly for $\tilde P^{(0)}$ and $\tilde P^{(1)}$!**

On the Slow Scale

↪ **Now $\sigma = f(Z_t)$ with $\delta \ll 1$:**

$$dZ_t = \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Lambda(Z_t)\right) dt + \sqrt{\delta} g(Z_t)\, dW_t^{(z)\star}$$

$$d\left\langle W^{(x)}, W^{(z)} \right\rangle = \rho_z \,.$$

↪ **Slow scale does not go away via averaging (rather "freezes"): $\tilde{P} = \tilde{P}(t,x,z)$, now we need:**

$$\tilde{P}_t - \tilde{P}_0 = \int_0^t \mathcal{L}_{BS}(f(Z_s))\tilde{P}_s \, ds + \left(\sqrt{\delta}\mathcal{M}_1(Z_s) + \delta \mathcal{M}_2(Z_s)\right) \tilde{P}_s$$
$$+ f(Z_s) X_s \partial_x \tilde{P}_s dW_s^{(x)} + \sqrt{\delta} g(Z_t) \partial_z \tilde{P}_s dW_s^{(z)} = \text{martingale} + \mathcal{O}(\varepsilon) \,,$$

with

$$\mathcal{M}_1 = g(z)\left(\rho_z f(z) x \partial^2_{xz} - \Lambda(z)\partial_z\right), \quad \mathcal{M}_2 = \frac{1}{2} g^2(z)\partial_z^2 + c(z)\partial_z \,.$$

$$\to \mathcal{L}_{BS}(\sigma(z))\tilde{P}_t + \sqrt{\delta}\mathcal{M}_1(z)\tilde{P}_t = 0 \,, \quad \tilde{P}_T = h \,.$$

Can solve explicitly for $\tilde{P}$ again.

Combined approximation:

$\hookrightarrow \sigma = \sigma(Y_t, Z_t)$.

- **Decomposition** $\tilde{P} = P^{(0)} + P^{(y)} + P^{(z)} + \mathcal{O}(\varepsilon)$.

$$\begin{aligned} \mathcal{L}_{BS}(\bar\sigma(z))(P^{(0)}) &= 0\,; & P^{(0)}|_T &= h\,, \\ \mathcal{L}_{BS}(\bar\sigma(z))(P^{(y)}) &= \mathcal{A}^\varepsilon P^{(0)}\,; & P^{(y)}|_T &= 0\,, \\ \mathcal{L}_{BS}(\bar\sigma(z))(P^{(z)}) &= -\left\langle \mathcal{M}_1^\delta \right\rangle_{\Phi_Y} P^{(0)}\,; & P^{(z)}|_T &= 0\,, \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= -V_2^\varepsilon(x^2\partial_x^2) - V_3^\varepsilon(x\partial_x)(x^2\partial_x^2)\,, \\ & \quad V_2^\varepsilon(z) = \frac{\sqrt{\varepsilon}}{2}\left\langle \beta\Lambda_y\phi' \right\rangle_{\Phi_Y}, V_3^\varepsilon(z) = -\frac{\sqrt{\varepsilon}\rho_y}{2}\left\langle \beta f\phi' \right\rangle_{\Phi_Y}, \\ \left\langle \mathcal{M}_1^\delta \right\rangle_{\Phi_Y} &= \sqrt{\delta}\rho_z g \langle f \rangle_{\Phi_Y} x\partial_{xz}^2 - \sqrt{\delta} g \langle \Lambda_z \rangle_{\Phi_Y} \partial_z\,, \\ &= V_1^\delta x\partial_{xz}^2 + V_0^\delta \partial_z\,. \end{aligned}$$

In practice: (i) $\partial_z \mapsto \bar\sigma'\partial_\sigma$. (ii) Eliminate V_2 via parameter reduction step.

Implied Volatility

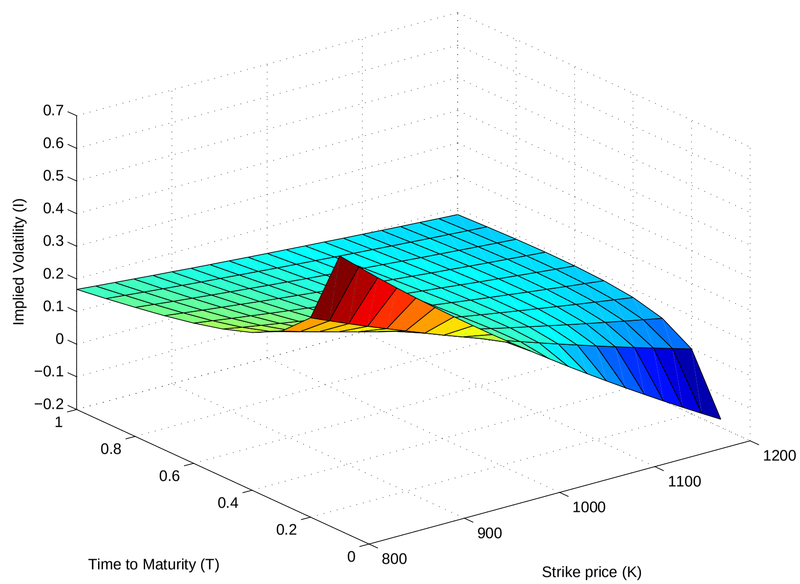
- Consider the situation of a call:

$\hookrightarrow$ The implied volatility I :

$$I \approx (b^\star + \tau b^\delta) + \left(a^\varepsilon + \tau\, a^\delta\right) \frac{\log K/X}{\tau},$$

coefficients $a^\varepsilon, b^\star, a^\delta, b^\delta$ affine functions of the V's.

$\hookrightarrow$ calibrate V's from implied volatility.

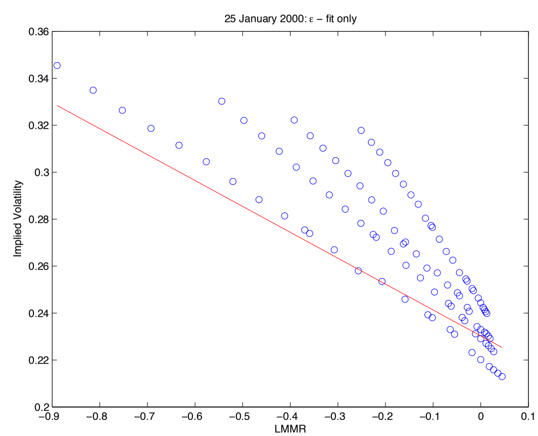


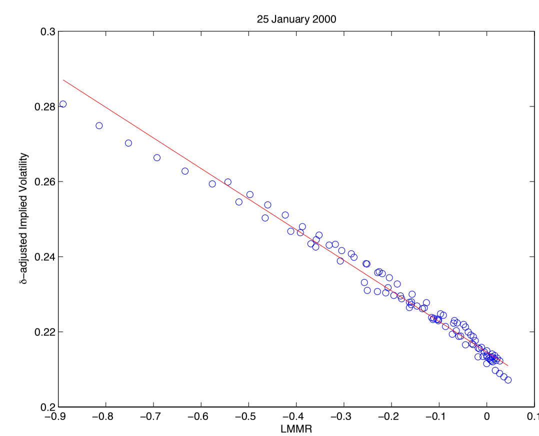
The implied volatility surface approximation for $(a^\varepsilon, b^\star, a^\delta, b^\delta) = (-.2, .2, -.015, -.08)$.
For a fixed time to maturity the surface is affine in $\log(K/x)$.

Long dated options and multiscale dynamics

$$
\begin{aligned}
dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(x)} \\
dY_t &= \frac{1}{\varepsilon}\alpha(Y_t)dt + \sqrt{\frac{1}{\varepsilon}}\beta(Y_t)dW_t^{(y)} \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^{(z)}
\end{aligned}
$$

with $d\left\langle W^{(x)}, W^{(y)} \right\rangle_t = \rho_y dt$, $d\left\langle W^{(x)}, W^{(z)} \right\rangle_t = \rho_z dt$ **and** $\varepsilon \ll T \ll \frac{1}{\delta}$.





Right : calibartion wrt $a^{\varepsilon}, b^{\star}$; **Left : calibartion wrt** $a^{\varepsilon}, b^{\star}, a^{\delta}, b^{\delta}$.

Fouque et al, SIAM MMS 03.

AN EXOTIC EXAMPLE AND *LINKAGE*

- An average strike option:

$$h_a = \left(X_T - \frac{1}{T} \int_0^T X_s ds \right)^+ ,$$

$$\hookrightarrow dI_t = X_t dt, \quad \mathcal{L}_2 \mapsto \widehat{\mathcal{L}_2} = \mathcal{L}_2 + x\partial_I .$$

- The price correction solves

$$\begin{aligned} < \widehat{\mathcal{L}_2} > \widehat{P}_1 &= \mathcal{L}_s \widehat{P}_0 , \\ \widehat{P}_1(T,x,v,I) &= 0, \end{aligned}$$

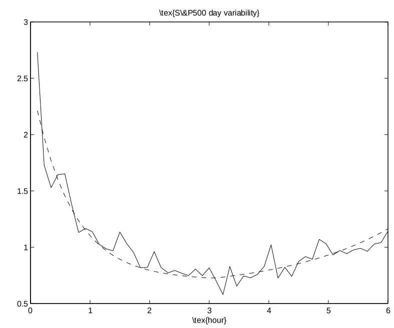
where

$$\mathcal{L}_s := \left\{ \mathcal{A}^\varepsilon - \left\langle \mathcal{M}^\delta \right\rangle_{\Phi_Y} \right\},$$

and $\mathcal{L}_s$ is determined by the calibrated effective market parameters $a^\varepsilon, b^\star, a^\delta, b^\delta$!

Approximation and Approach Stability

$\hookrightarrow$ Example Periodicity :Volatility as function of trading hour :



$$dX_t \;=\; \mu X_t dt + f(Y_t, t/\varepsilon) X_t dW_t^{(x)}\,.$$

• Averaging functional $\langle\cdot\rangle_{\Phi_Y}$ modified to include averaging over periodic cycle. • The structure of the approximation is not changed only the **parameter interpretation** is.

$\hookrightarrow$ Jump component in fast scale; again modified averaging with respect invariant distribution of the jump process.

$\hookrightarrow$ Non Markovian models; use "conditional shift"

$$\phi(Y_t) \mapsto -\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^T \left(f^2(Y_s) - \bar{\sigma}^2\right) ds \mid \mathcal{F}_t\right]\,.$$

Some **References**

Derivatives in financial markets with stochastic volatility; *Fouque, Papanicolaou & Sircar; Cambridge 2000*.

Singular perturbation in option pricing; *Fouque, Papanicaolaou, Sircar & Solna; SIAP 2003*.

Multiscale stochastic volatility asymptotics; *Fouque, Papanicaolaou, Sircar & Solna; SIAM MMS 2003*.

On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility; *Alos, Leon & Vives; Finance Stoch 2007*.

Asymptotic Analysis for Stochastic Volatility: Martingale Expansion; *Fukasawa; preprint 2010*.

NEXT: PARAMETER INTERPRETATION

- **Model under risk neutral :**

$$
\begin{aligned}
dX_t &= rX_t dt + f(Y_t) X_t dW_t^{(x)\star} \\
dY_t &= \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda(Y_t) \right) dt + \beta(Y_t) dW_t^{(y)\star},
\end{aligned}
$$

where the Brownian motions $W^{(*)\star}$ have covariation ρ_y and Λ is a combined market price of risk parameter.

- **Absorption :** $u(t,s) = e^{rt}\tilde{P}(T-t, e^s) + \mathcal{O}(\varepsilon)$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \tilde{\mu}\frac{\partial u}{\partial s} + \breve{\sigma}^2 \frac{\partial^2 u}{\partial s^2} + \mathrm{sign}(\rho) v \frac{\partial^3 u}{\partial s^3}, \\
&u(0,s) = e^{-rT} h(e^s) := \tilde{h}(s), \\
&\tilde{\mu} = r - \frac{\bar{\sigma}^2}{2} + V_2^\varepsilon \qquad \breve{\sigma}^2 = \frac{\bar{\sigma}^2}{2} + (V_2^\varepsilon - V_3^\varepsilon) > 0 \\
&v = |V_3^\varepsilon| \neq 0 .
\end{aligned}
$$

An Old Paper by Krylov

- **Krylov considered:**

$$\frac{\partial u}{\partial t} = (-1)^{q+1} \frac{\partial^{2q} u}{\partial s^{2q}}.$$

• Wiener measure for $q = 1$. • In general define a signed measure on path space. • Deduced Feynman-Kac and used this for deducing a type of "arc-sine law".

- **Orsingher considered:**

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial s^3},$$

whose fundamental solution gives a non-symmetric signed measure and can similarly be associated with "Orsinger canonical" pseudo processes. • Faynman-Kac still valid. • Deduced an "asymmetric version" of the arc-sine law, time epochs of negative path values partly cancels.

Pseudo-Process Representation

Let $(B_t)_{0\leq t\leq T}$ be a Brownian motion and $(X_t)_{0\leq t\leq T}$ a pseudo-process defined on a signed probability space $(\Omega^*, \mathcal{F}^*, P^*)$ (sum of Orsingher canonical process and Brownian motion on product space) with density:

$$p(t,x) = \frac{1}{\sqrt[3]{3vt}}\mathbb{E}\left[Ai\left(\frac{-(x+\tilde{\mu}t)\mathrm{sign}(\rho) - \sqrt{2\breve{\sigma}^2}B_t}{\sqrt[3]{3vt}}\right)\right],$$

for all $0 < t \leq T$ and $p(0,.) = \delta_0$, such that for all $0 \leq t \leq T$,

$$u(t,x) = \mathbb{E}^*\Big[\tilde{h}(x+X_t)\Big],$$

where $\mathbb{E}^*$ denotes the expectation (the integral) with respect to the signed measure and Ai is the Airy function.

For fixed $(t,x) \in [0,T] \times [0,+\infty[$, the price of the European option is given by

$$P^\epsilon(t,x) \overset{\epsilon\downarrow 0}{\sim} e^{-r(T-t)}\mathbb{E}^*\Big[h(log(x+X_{T-t}))\Big].$$

Standardized Pseudo-Density

- **Pseudo-density** $p\left(t, -\left(s\sqrt{2\breve{\sigma}^2 t}\,\text{sign}(\rho) + \tilde{\mu} t\right)\right) = q(x; \Theta(t))$:

$$q(x;\Theta) = \frac{1}{\Theta}\left[Ai\left(\frac{\cdot}{\Theta}\right) \star \mathcal{N}(\cdot)\right](x)\,, \quad \int_{-\infty}^{\infty} q(x;\Theta)\,dx = 1\,, \quad \Theta = \frac{\sqrt[3]{3v}}{\sqrt{2\breve{\sigma}^2}\,\sqrt[6]{t}}.$$

- **Explicit form:**

$$q(x;\Theta) = \exp\left(\frac{1}{12\Theta^6} + \frac{x}{2\Theta^3}\right)\Theta^{-1} Ai\left(\frac{x}{\Theta} + \frac{1}{4\Theta^4}\right),$$

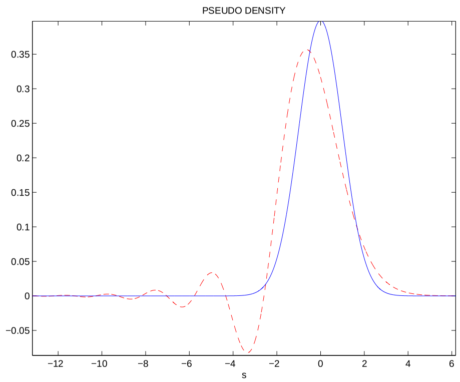
- Gaussian limit: $q(x;\Theta) \overset{\Theta \to 0}{\sim} \mathcal{N}(x)$. • Small volatility and times gives relatively large support of Airy function and strong skew.
- For smooth payoff function $\tilde{h}$ the support of Airy function relatively largest for $t^\star = 1/\breve{\sigma}^2$. **Tails:**

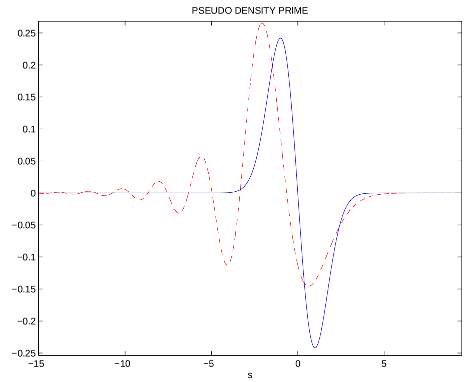
$$q(x;\Theta) \overset{x \to \infty}{\sim} \frac{\exp\left(\frac{1}{12\Theta^6}\right)}{\left(\sqrt[4]{x}\Theta^{3/4} 2\sqrt{\pi}\right)} \exp\left(\frac{-2\left(x^{3/2} - 3x/(4\Theta^{3/2})\right)}{3\Theta^{3/2}}\right),$$

$$q(x;\Theta) \overset{x \to -\infty}{\sim} \frac{\exp\left(\frac{1}{12\Theta^6}\right)}{\left(\sqrt[4]{x}\Theta^{3/4}\sqrt{\pi}\right)} \exp\left(\frac{-|x|}{2\Theta^3}\right) \sin\left(\frac{2}{3}\left(\frac{x}{\Theta} + \frac{1}{4\Theta^4}\right) + \frac{\pi}{4}\right).$$

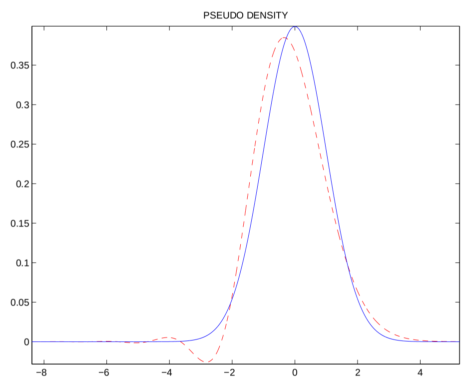
Pseudo-Density :

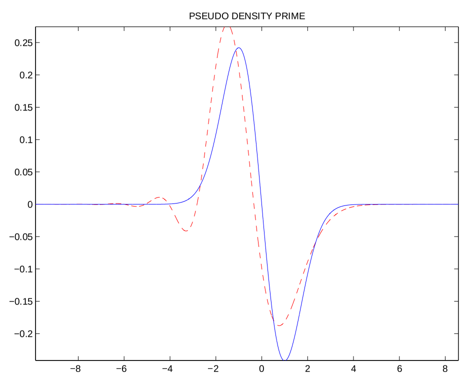
Airy dominated : $\Theta = 1$ ($v = 10^{-3}/3, \breve{\sigma}^2 = .005, t = 1$).

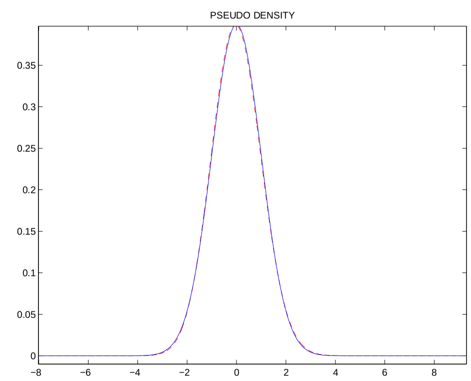


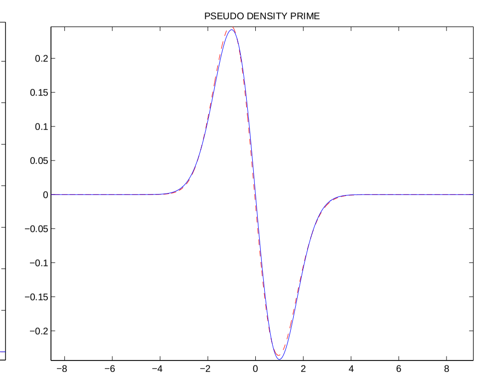


(left) Transition Zone : $\Theta = .7$ ($v = 10^{-3}/3, \breve{\sigma}^2 = .005, t \approx 3$).
(right) Gaussian Limit : $\Theta = .33$ ($v = 10^{-3}/3, \breve{\sigma}^2 = .005, t \approx 700$).









$\hookrightarrow$ Changing sign of correlation corresponds to time-reversal of the Airy function & pseudo-density.

FINALLY: Connection to Credit Risk; Intensity Based Modeling

◇ **We aim at COMPUTING:**

I: The joint survival probabilities:

$$q_n(T) = \mathbb{E}^\star \left\{ e^{-\int_0^T (\lambda_{1s} + \cdots + \lambda_{ns})\, ds} \right\}$$

for $n = 1, \cdots, N$.

II: The loss distribution:

$$p_n(T) = \mathbb{P}^\star \{ (\#\textbf{names defaulted by time } T) = n \} .$$

Foueque et al, MMS 2007.

Stochastic Volatility Multiname Gaussian Model

• Choose exchangable models and let the volatility be driven by common fast mean reverting SV factor:

$$\begin{aligned} dX_t^{(i)} &= \kappa(\theta - X_t^{(i)})dt + \sigma(Y_t)\, dW_t^{(i)}, \quad \text{with} \quad X_0^{(i)} = x\,, \\ dY_t &= \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\, dW_t^{(y)}, \quad \lambda_{it} = X_t^{(i)}\,. \end{aligned}$$

where ε is the natural time scale/mean reversion time for the volatility factor.

• name-name correlations:

$$d\left\langle W^{(i)}, W^{(j)} \right\rangle_t = \rho_X\, dt\,, \quad i \neq j\,,$$

& name-volatility correlations:

$$d\left\langle W^{(i)}, W^{(y)} \right\rangle_t = \rho_Y\, dt\,.$$

$\hookrightarrow$ Negative intensities? Duffie-Singleton, Credit Risk 2003: *“the computational advantage with explicit solutions may be worth the approximation error associated with this Gaussian formulation”*.

The Corrected Multiname Survival Probability

For

$$q_n(T) \;=\; \mathbb{E}\left\{ e^{-\int_0^T (X_s^{(1)}+\cdots+X_s^{(n)})\,ds} \mid X_0^{(1)} = x, \cdots, X_0^{(n)} = x, Y_0 = y \right\}$$

we have in $\varepsilon \to 0$ limit:

$$q_n(T) \sim \exp\left(\sqrt{\varepsilon}\rho_Y v_3 B^{(3)}(T)(n^2 + n^2(n-1)\rho_X) \right)$$
$$\times \exp\left(-n \left[\theta_\infty (T - B(T)) + [1 + (n-1)\rho_X]\bar{\sigma}^2 B^2(T)/(4\kappa) + xB(T) \right] \right)$$

with

$$B(T) = \frac{1 - \exp(-\kappa T)}{\kappa}, \qquad B^{(3)}(T) = \int_0^T B(t)^3\,dt,$$

$$\theta_\infty = \theta - [1 + (n-1)\rho_X]\frac{\bar{\sigma}^2}{2\kappa^2}, \qquad \bar{\sigma}^2 = \left\langle \sigma^2(y) \right\rangle.$$

$\rightarrow$ **Combined correlation gives strong correlation gearing.**

Leading Loss Distribution Symmetric Case

- **Loss distribution:** $p_n = \binom{N}{n} \sum_{j=0}^{n} \binom{n}{j} q_{N+j-n} (-1)^j$,

Conditioning:

$$q_n = \mathbb{E}^* \left\{ e^{-d_1 n + d_2 n X} \right\}, \quad p_n = \mathbb{E}^* \left\{ \tilde{p}_n (d_1 - d_2 X) \right\},$$

for $\tilde{p}_n(d)$ the binomial loss distribution with $q_n = \exp(-dn)$ and X the pseudo-process ($v_3 > 0$).

$\hookrightarrow$ Loss distribution is found via integration of binomial distribution with respect to an in general signed and non-symmetric measure. Gives loss distribution to $\mathcal{O}(\varepsilon)$.

- **Name Heterogeneity One volatility driving factor, heterogeneous level and volatility:** $dX_{it} = \kappa(\theta_i - X_t^{(i)}) dt + v_i f(Y_t)\, dW_t^{(i)}$, **for $1 \le i \le N$ and with v_i ad θ_i constants.**

$\hookrightarrow$ **Constant volatility survival probability with name-name correlation and for X Gaussian :**

$$q(T; \mathbf{x}, n) = \mathbb{E} \left\{ \prod_{i=1}^{n} e^{X \sigma_i \sqrt{\rho_X B^{(2)}(T)}} \tilde{A}_i(T) e^{-B(T) x_i} \right\} = \mathbb{E} \left\{ \prod_{i=1}^{n} \tilde{q}_i(T; x_i) \right\}$$

$\rightarrow$ Computation via Hull-White algorithm for independent case. $\hookrightarrow$ General case X modeled as pseudo-variable.

CONCLUSION and FURTHER ISSUES

- Multiscale Stochastic Volatility provides flexible and parsimonious model class extensions that can be dealt with using singular and regular perturbation methods. Derives from a consistent framework honoring underlying price dynamics and provides a linkage between derivatives facilitating calibration.

- "Model idependent" approach to dealing with uncertain and changing parameters.

- Situations with analytic constant parameter solutions in particular feasible.

- Generalizations to structural models, hedging schemes, indifference pricing, free boundary value problems, other markets ....

- Important issues: – "identifiability" of model parameters, efficient computations...