An Improved Procedure for VaR/CVaR Estimation under Stochastic Volatility Models

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Outline

- Risk Management in Practice: Value at Risk (VaR) / Conditional Value at Risk (CVaR)
- Volatility Estimation: Corrected Fourier Transform Method
- Estimate Extreme Probability by Efficient Importance Sampling
- Backtesting for VaR Estimation

Value at Risk

- Let $r(t)$ be an asset return at time t. Its $\alpha \times 100\%$ VaR, denoted by VaR_α , is defined by the $(1-\alpha)\times 100\%$ percentile of $r(t)$. That is,

$$P\left(r(t) \le VaR_\alpha\right) = 1-\alpha$$

That is a risk controller has a $\alpha \times 100\%$ confidence that the asset price will not drop below VaR_α at time t.

Aspects about VaR

- Mathematically, it is not a coherent risk measure* because it doesn't satisfy the risk diversification principal. Instead, CVaR does!.
- Practically, VaR is commonly required by financial regulations (Basel Ⅱ Accord).

* Artzner P., F. Delbaen, J.-M. Eber, and D. Heath, "Coherent Measures of Risk," *Mathematical Finance*, 9 (1999): 203-28.

Estimation of VaR

- Riskmetrics: normal assumption under EWMA model.
- Historical Simulation: generate scenarios
- Model Dependent Approach: Discrete-Time vs. Continuous-Time Models

A Nonparametric Method to Estimate Volatility: Fourier Transform Method*

- Assume a diffusion process

$$du(t) = \mu(t)\,dt + \sigma(t)\,dW_t,$$

- Task: Given return time series $u(t)$, estimate the volatility $\sigma(t)$.

* Malliavin and Mancino(2002, 2009)

Fourier Transform Method(Step 1)

- Compute the Fourier coefficients of du by

$$a_0(du) = \frac{1}{2\pi}\int_0^{2\pi} du(t),$$

$$a_k(du) = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)\, du(t),$$

$$b_k(du) = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)\, du(t).$$

- Then,

$$u(t) = a_0 + \sum_{k=1}^{\infty}\left[-\frac{b_k(du)}{k}\cos(kt) + \frac{a_k(du)}{k}\sin(kt)\right].$$

Fourier Transform Method(Step 2)

- Fourier coefficients of variance $\sigma^2(t)$,

$$a_k\left(\sigma^2\right)=\lim_{N\to\infty}\frac{\pi}{2N+1}\sum_{s=-N}^{N-k}\left[a_s^*\left(du\right)a_{s+k}^*\left(du\right)+b_s^*\left(du\right)b_{s+k}^*\left(du\right)\right],$$

$$b_k\left(\sigma^2\right)=\lim_{N\to\infty}\frac{\pi}{2N+1}\sum_{s=-N}^{N-k}\left[a_s^*\left(du\right)b_{s+k}^*\left(du\right)-b_s^*\left(du\right)a_{s+k}^*\left(du\right)\right],$$

where n_0 is any positive integer so that

$$\sigma_N^2\left(t\right)=\sum_{k=0}^{N}\left[a_k\left(\sigma^2\right)\cos\left(kt\right)+a_k\left(\sigma^2\right)\sin\left(kt\right)\right].$$

Fourier Transform Method(Step 3)

Reconstruct the time series variance $\sigma^2(t)$.

- Finally, $\sigma_N^2(t)$ is an approximation of $\sigma^2(t)$ as N approaches infinity, which can be given by classical Fourier-Fejer inversion formula.

$$\sigma^2(t) = \lim_{N\to\infty} \sigma_N^2(t) \quad \text{in prob.}$$

Smoothing

- We add a function into the final computation of time series variance in order to smooth it.

$$\sigma^2(t) = \lim_{N\to\infty} \sum_{k=0}^{N} \varphi(\delta k)\left[a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt)\right],$$

  where $\varphi(x) = \dfrac{\sin^2(x)}{x^2}$ is a function in order to smooth the trajectory and δ is a smoothing parameter.
- Reno (2008) alerts the boundary effect in the Fourier transform method.

A Price Correction Scheme: First Order

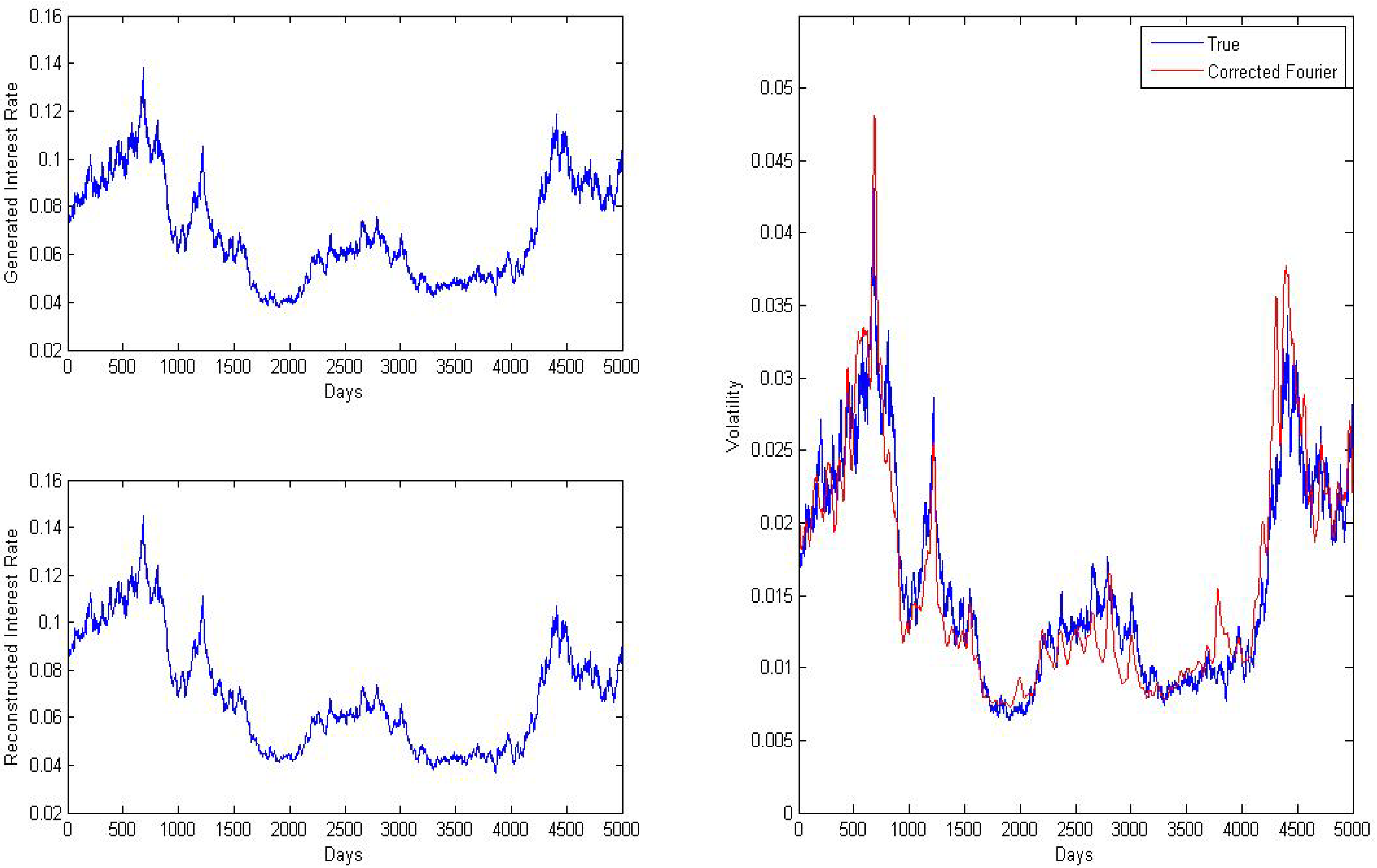
- Idea: (Nonlinear) Least Squares Method for first-order correction

$$r_t \approx \sigma_t \delta_t \varepsilon_t$$
$$\approx \exp\left(\left(a + b\hat{Y}_t\right)/2\right)\delta_t \varepsilon_t .$$

- Then by MLE to regress out a and b

$$\boxed{\ln\left(\frac{r_t}{\delta_t}\right)^2 = a + b\hat{Y}_t + \ln \varepsilon_t^2 .}$$

Simulation Study – Local Volatility



Stochastic Volatility Model Estimation

- Assuming that the driving volatility process is governed by the Ornstein-Uhlenbeck process,

$$dY_t = \alpha\left(m - Y_t\right)dt + \beta dW_t. \qquad (1)$$

- We use the corrected estimator $a + b\hat{Y}_t$ to further estimate model parameters (α, β, m) of $\hat{Y}_t$ by means of maximum likelihood method.

Stochastic Volatility Model Estimation (cont.)

- For a given set of observations $\left(Y_1, Y_2, \ldots, Y_N\right)$ the likelihood function is

$$L(\alpha, \beta, m) = \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi\beta^2\Delta_t}} \exp\left\{ -\frac{1}{2\beta^2\Delta_t}\left[Y_{t+1} - \left(\alpha m \Delta_t + (1 - \alpha\Delta_t) Y_t \right) \right]^2 \right\},$$

where Δ_t denotes the length of discretized time interval.

Stochastic Volatility Model Estimation (cont.)

- By maximizing the right hand side over the parameters (α, β, m), we obtain the following maximum likelihood estimators

$$\hat{\alpha} = \frac{1}{\Delta_t}\left[1 - \frac{\left(\sum_{t=2}^{N} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t\right) - (N-1)\left(\sum_{t=1}^{N-1} Y_t Y_{t+1}\right)}{\left(\sum_{t=1}^{N-1} Y_t\right)^2 - (N-1)\left(\sum_{t=1}^{N-1} Y_t^2\right)}\right], \tag{3}$$

$$\hat{\beta} = \sqrt{\frac{1}{N\Delta_t}\sum_{t=1}^{N-1}\left[Y_{t+1} - \left(\alpha m \Delta_t + (1-\alpha\Delta_t) Y_t\right)\right]^2}, \tag{4}$$

$$\hat{\mathrm{m}} = \frac{-1}{\hat{\alpha}\Delta_t}\left[\frac{\left(\sum_{t=2}^{N} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t^2\right) - \left(\sum_{t=1}^{N-1} Y_t\right)\left(\sum_{t=1}^{N-1} Y_t Y_{t+1}\right)}{\left(\sum_{t=1}^{N-1} Y_t\right)^2 - (N-1)\left(\sum_{t=1}^{N-1} Y_t^2\right)}\right], \tag{5}$$

Simulation Study – Stochastic Vol

- Let the stochastic volatility model

$$\begin{cases} dS_t = \mu S_t dt + \exp\left(Y_t/2\right) S_t dW_{1t}, \\ dY_t = \alpha\left(m - Y_t\right) dt + \beta dW_{2t}. \end{cases}$$

- To empirically test our price correction scheme, we set model parameters as follows:

$$\mu = 0.01,\ S_0 = 50, Y_0 = -2,\ m = -2,\ \alpha = 5,\ \beta = 1,$$

and with the discretization length $\Delta_{\mathrm{t}} = 1/5000$ so as to generate volatility series $\sigma_t = \exp\left(Y_t/2\right)$ and asset price series S_t .

Simulation Study(cont.)

- Two criteria are used for performance comparison: Mean squared errors (MSE) and Maximum absolute errors (MAE).
- Comparison results are shown below:

	Fourier method	Corrected Fourier method
Mean squared error	0.0324	0.0025
Maximum absolute error	0.3504	0.1563

Estimate Extreme Probability

- Given a Markovian dynamic model of an asset price S_t , its return process is $r_T = \ln\left(S_T / S_0\right)$.

- Given a loss threshold D, the extreme probability is defined by

$$P\left(0, S_0; D\right) = E\left[I\left(r_T \leq D\right) \middle| S_0\right].$$

Note: solve VaR_α from $P\left(0, S_0; VaR_\alpha\right) = 1 - \alpha$.

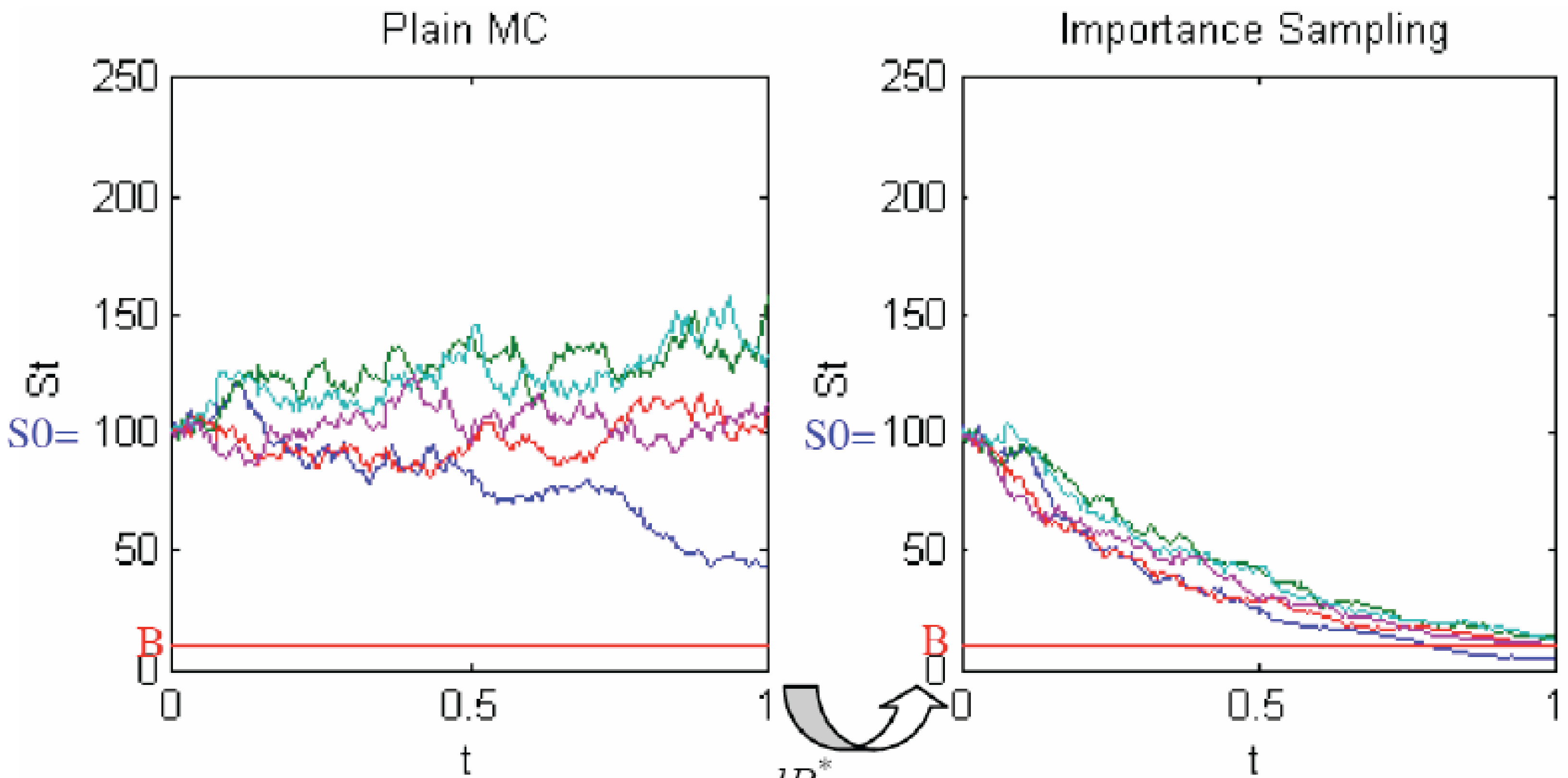
- $CVaR = E\left[r_T \middle| r_T \leq VaR_\alpha\right]$. (Expected Shortfall)

Importance Sampling

- Given the Black-Scholes Model under measure P, choose $\frac{dP}{d\tilde{P}} = Q_T(h) = \exp\left(h\tilde{W}_T - \frac{h^2 T}{2}\right)$ satisfing $\tilde{E}[S_T] = S_0 \exp(D)$.
- Then $h = \frac{\mu}{\sigma} - \frac{D}{\sigma T}$, the extreme probability becomes $P(0, S_0) = \tilde{E}\left[\mathrm{I}(r_T \le D) Q_T(h) \middle| S_0\right]$. (6)
- The unbiased importance sampling estimator of $P(0, S_0)$ is $\frac{1}{N}\sum_{i=1}^{N} \mathrm{I}\left(r_T^{(i)} \le D\right) Q_T^{(i)}(h)$. (7)

Trajectories under different measures: Black-Scholes Model

Simulation of the stock price :



$$\frac{dP^*}{d\widetilde{P}} = Q_T$$

Importance Sampling(cont.)

- **Theorem:**
  Under the Black-Scholes model, the proposed importance sampling estimator is asymptotically optimal or efficient under some scaling scenarios in time and space.

  Proof: The variance rate of the proposed importance sampling scheme approaches zero.

Importance Sampling under Stochastic Volatility Model

- Stochastic volatility model:

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t dW_{1t} \\ \sigma_t = \exp\left(Y_t/2\right) \\ dY_t = \alpha\left(m - Y_t\right)dt + \beta dW_{2t} \end{cases}$$

- Ergodic property of the averaged variance process

$$\frac{1}{T}\int_0^T f(Y_t^\varepsilon)^2 dt \overset{a.s.}{\rightarrow} \bar{\sigma}^2, \text{for } \varepsilon \rightarrow 0$$

where ε denotes a small time scale and Y_t^ε denotes a fast mean-reverting process.

Importance Sampling under Stochastic Volatility Model(cont.)

- E.g. $\alpha = \frac{1}{\varepsilon}, \beta = \sqrt{\frac{2\nu}{\varepsilon}}.$
- So that the importance sampling as forementioned can be applied.
- CVaR estimation can be easily solved.

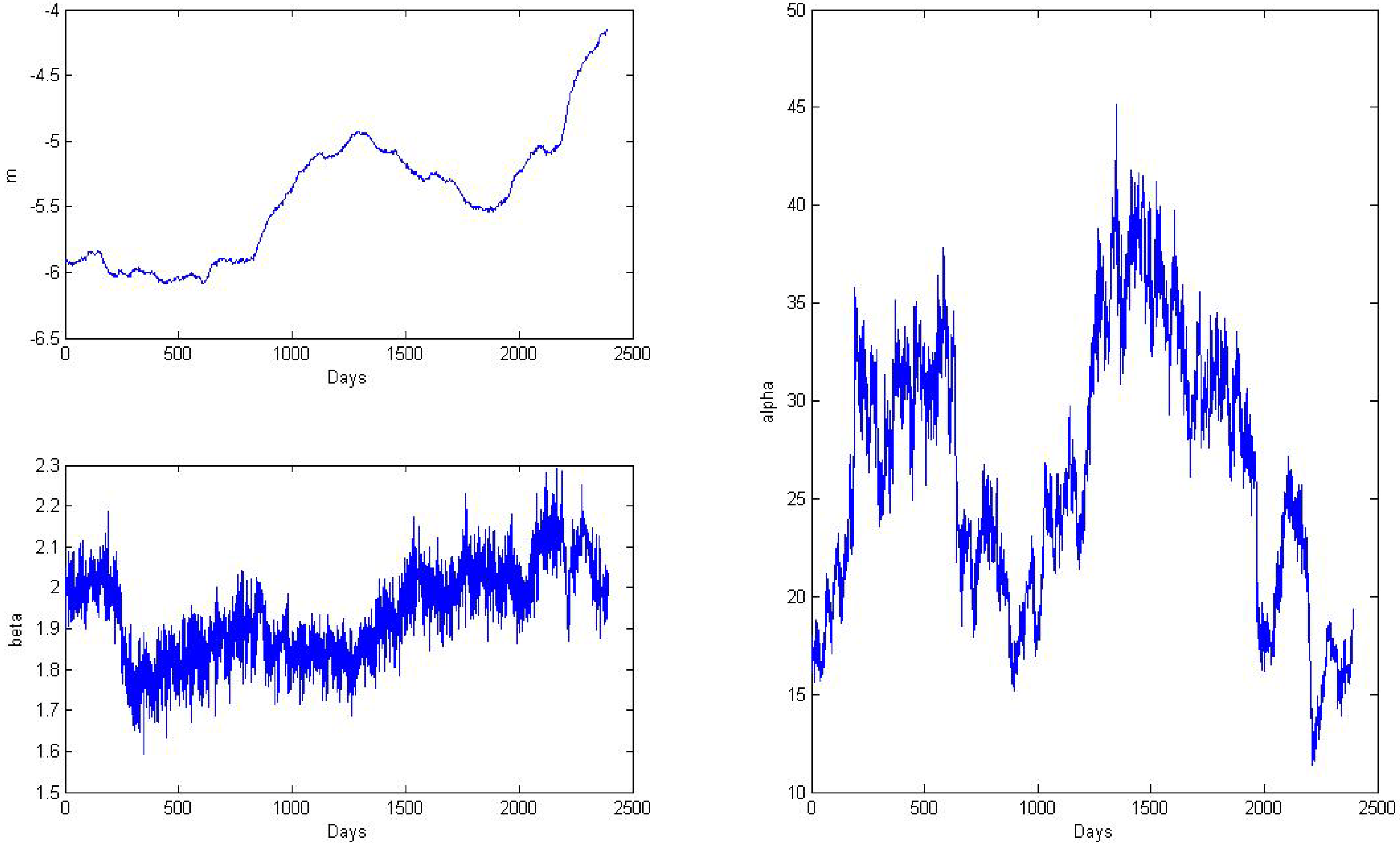
Numerical Results of VaR/CVaR Estimation

- Given model parameters of stochastic volatility: $m=-5, \alpha=5, \beta=1, S_0=50, Y_0=-3, \mu=0$

ρ	$c = VaR_{99\%}$	CVaR	
		N. Approx.	IS
0.8	-0.0339	-0.0347	-0.0386 (7.5073E-05)
0.4	-0.0335	-0.0343	-0.0378 (7.3833E-05)
0	-0.0323	-0.0331	-0.0367 (7.2498E-05)
-0.4	-0.0317	-0.0325	-0.0366 (7.2739E-05)
-0.8	-0.0310	-0.0319	-0.0351 (6.9643E-05)

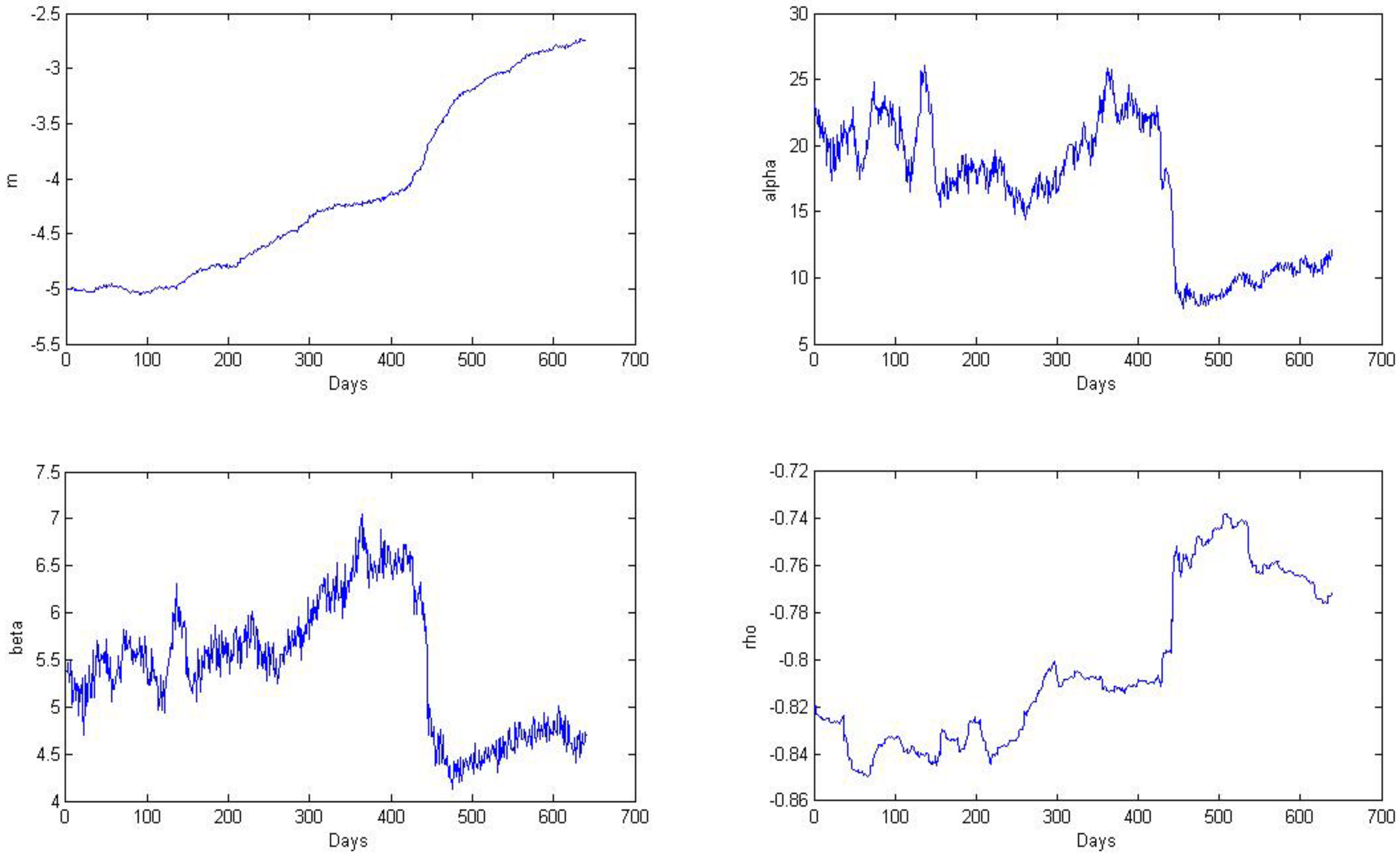
SV Model Estimation: CAD/USD

- Data sample period: 1998.01.05-2009.07.24



SV Model Estimation: S&P500 / VIX

- Data sample period: 2005.01.03-2009.07.24



Backtesting Outcomes of S&P 500 VaR Estimate

Data sample period: 2005.01.03-2009.07.24

RiskMetrics			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	Reject VaR Model	LRind	**Don't Reject VaR Model**
LRcc	Reject VaR Model	LRcc	Reject VaR Model
Historical Simulation			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	**Don't Reject VaR Model**
LRcc	Reject VaR Model	LRcc	Reject VaR Model
SV			
Significance	1%	Significance	5%
LRuc	**Don't Reject VaR Model**	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	**Don't Reject VaR Model**
LRcc	**Don't Reject VaR Model**	LRcc	Reject VaR Model
GARCH(1,1)			
Significance	1%	Significance	5%
LRuc	Reject VaR Model	LRuc	Reject VaR Model
LRind	**Don't Reject VaR Model**	LRind	Reject VaR Model
LRcc	Reject VaR Model	LRcc	Reject VaR Model

Conclusion

- Remove boundary effect of Fourier transform method for volatility estimation.
- (efficient) importance sampling methods are investigated.
- VaR backtesting results for FX and equity data.

Thank You