Market indifference prices

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Asset price models

Mathematical Finance:

- price dynamics **exogenous**: semimartingale models
- stochastic analysis

+ mathematically tractable
+ dynamic model: hedging
+ 'easy' to calibrate: volatility

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Economics:

- prices **endogeneous**: demand matches supply
- equilibrium theory
- \+ undeniably reasonable explanation for price formation
- \+ excellent qualitative properties
- – difficult to calibrate: preferences, endowments
- – quantitative accuracy?

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Our goal:

Bridge the gap between these price formation principles!

Utility based prices

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Derive dependence of prices on demand.

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- Davis price or marginal utility indifference price:

$$p = \frac{\partial}{\partial q}\bigg|_{q=0} x(q) = \frac{\mathbb{E}u'(\alpha + V_T(Q^0))\psi}{\mathbb{E}u'(\alpha + V_T(Q^0))} = \mathbb{E}^0\psi$$

Market indifference prices

- We need to be able to analyze quoted prices in a dynamic setting:
  - $\alpha \neq 0$, $q \neq 0$ in general: indifference prices hard to compute
  - work conditionally on $\mathscr{F}_t$ $(0 \leq t \leq T)$
  - formidable technical difficulties: optimal investment strategies have to be determined, processes such as conditional indirect utilities must be shown to have good versions and we need to understand the dependence of their martingale characteristics on q ...

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- Market indifference prices address these issues:
  - (almost) as easy to compute as certainty equivalents: convex duality of saddle functions
  - hedging replaced by formation of Pareto allocation for market makers' endowments
  - no more *optimal* control, static concept

General setting

Financial model

- beliefs and information flow described by stochastic basis $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$
- marketed claims: European with payoff profiles $\psi_i \in L^0(\mathscr{F}_T)$ $(i = 1, \ldots, I)$ possessing all exponential moments
- utility functions $u_m : \mathbb{R} \to \mathbb{R}$ $(m = 1, \ldots, M)$ with bounded absoulte risk aversion:

$$0 < c_* \leq -\frac{u_m''(x)}{u_m'(x)} \leq c^* < \infty$$

  $\rightsquigarrow$ similar to exponential utilities
- initial endowments $\alpha_0^m \in L^0(\mathscr{F}_T)$ $(m = 1, \ldots, M)$ have finite exponential moments and form a Pareto-optimal allocation

Pareto-optimal allocations

Recall:

- $\alpha = (\alpha^m) \in L^0(\mathscr{F}_T, \mathbb{R}^M)$ is **Pareto-optimal** if $\Sigma = \Sigma_m \alpha^m$ cannot be re-distributed to form a better allocation $\tilde{\alpha} = (\tilde{\alpha}^m)$:

  $$\mathbb{E}u_m(\tilde{\alpha}^m) \geq \mathbb{E}u_m(\alpha^m) \quad \text{with '$>$' for some } m \in \{1, \dots, M\} \quad .$$

- Pareto-optimal allocations realized through trades among market makers $\rightsquigarrow$ complete OTC-market

Characterizations of Pareto optima

Lemma

Equivalent for an allocation (α_m) with $\Sigma = \sum_m \alpha_m$:

(i) (α_m) is Pareto optimal.

(ii) Given the respective endowments $\tilde{e}_a$ $(a \in \mathscr{A})$ all agents will quote the same marginal indifference prices:

$$\Pi(X) = \frac{\mathbb{E}u'_m(\alpha^m)X}{\mathbb{E}u'_m(\alpha^m)} = \frac{\mathbb{E}u'_{\tilde{m}}(\alpha^{\tilde{m}})X}{\mathbb{E}u'_{\tilde{m}}(\alpha^{\tilde{m}})} \; (X \in L^\infty) \text{ for any } m, \tilde{m}\,.$$

(iii) (α^m) is the solution to a social welfare problem:

$$\sum_m w^m \mathbb{E}u_m(\alpha^m) \to \max \text{ subject to } \Sigma = \sum_m \alpha^m$$

for suitable weights $w^m > 0$ with $\sum_m w^m = 1$.

There are 1-1 correspondences: $w \leftrightarrow \Pi \leftrightarrow \alpha$

A single transaction

- pre-transaction endowment of market makers: $\alpha = (\alpha^m)$ with total endowment $\Sigma = \sum_m \alpha^m$
- investor submits passes $q = (q^1, \ldots, q^I)$ claims on to the market makers along with a cash transfer of size x
- total endowment of market makers after transaction

$$\tilde{\Sigma} = \Sigma + (x + \langle q, \psi \rangle)$$

  is redistributed among the market makers to form a new Pareto optimal allocation of endowments $\tilde{\alpha} = (\tilde{\alpha}^m)$

Obvious question:

How exactly to determine the cash transfer x and the new allocation $\tilde{\alpha}$?

A single transaction

Theorem

There exists a unique cash transfer $x = x(q)$ *and a unique Pareto-optimal allocation* $\tilde{\alpha} = (\tilde{\alpha}^m(q))$ *of the total endowment* $\tilde{\Sigma}(x, q) = \Sigma + (x + \langle q, \psi \rangle)$ *such that each market maker is as well-off after the transaction as he was before:*

$$\mathbb{E}u_m(\tilde{\alpha}^m) = \mathbb{E}u_m(\alpha^m) \quad (m = 1, \ldots, M)\,.$$

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Note:

The cash transfer x can be viewed as the **market's indifference price** for the transaction q: it is the minimal amount for which the market makers can accommodate the investor's order without anyone of them being worse-off.

$\rightsquigarrow$ most friendly market environment for our investor!

Basic questions about market indifference prices

- How does the market indifference price depend on the transaction's size?
- Under what conditions is there a liquidity premium?
- What are its key determinants?
- How does the market's pre-transaction exposure affect the market indifference price?
- How to take into account the market makers' risk aversion and ability to hedge?
- Is there a difference between a model with several market makers and one with a representative market maker?
- ...

Expansions of market indifference prices

Theorem

The indifference price $x = x(q)$ is C^2 with

$$x(q+\Delta q) - x(q) = -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi \rangle] + \frac{1}{2R_0}\mathbb{E}_{\mathbb{R}}[(\langle \Delta q, \psi \rangle - \mathbb{E}_{\mathbb{Q}}\langle \Delta q, \psi \rangle)^2] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 var_{\rho}[Z\Delta q]\right] + o(|\Delta q|^2), \quad \Delta q \to 0,$$

where

- $\mathbb{Q} \sim \mathbb{P}$ *is the equilibrium pricing measure determined by the market makers' Pareto allocation*
- R_0 *is the market's risk tolerance at transaction time*
- $\mathbb{R} \sim \mathbb{Q}$ *is the market's risk tolerance measure*
- ρ *is the vector of the market makers' relative risk tolerances*
- Z *describes the sensitivities of Pareto weights w.r.t.* q

Some observations

$$
\begin{aligned}
x(q+\Delta q) - x(q) &= -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi \rangle] \\
&\quad + \frac{1}{2R_0}\mathbb{E}_{\mathbb{R}}[(\langle \Delta q, \psi \rangle - \mathbb{E}_{\mathbb{Q}}\langle \Delta q, \psi \rangle)^2] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \mathrm{var}_{\rho}[Z\Delta q]\right] \\
&\quad + o(|\Delta q|^2), \quad \Delta q \to 0,
\end{aligned}
$$

- Up to 1st order, the transaction costs are as in a small investor setting with pricing measure $\mathbb{Q}$.
- The market indifference price is convex in the transaction size.
- The liquidity premium is always nonnegative and vanishes if and only if we have a pure (and pointless) cash transaction: $\langle \Delta q, \psi \rangle \equiv$ const

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\end{aligned}
$$

- The liquidity premium splits into an aggregate component and one featuring the relative risk tolerances $\rho^m = R^m / \sum_l R^l$.
- Up to 2nd order, there is no difference between our multiple market maker model and a representative market maker model if and only if

$$
\mathbb{E}_{\mathbb{R}^l}\psi = \mathbb{E}_{\mathbb{R}^m}\psi \quad (l, m = 1, \ldots, M)
$$

  where $\mathbb{R}^m$ is market maker m's risk tolerance measure, i.e., if and only if the extra endowment with any tradable claim has the same 2nd order impact on every market maker's expected utility.

Key tool: Convex duality of saddle functions

Theorem

The representative agent's utility

$$r(v, x, q) = \max_{\alpha : \sum_m \alpha^m = \Sigma + (x + \langle q, \psi \rangle)} \sum_m v^m \mathbb{E} u_m(\alpha^m)$$

has the dual

$$\tilde{r}(u, y, q) = \sup_v \inf_x \{\langle v, u \rangle + xy - r(v, x, q)\}$$

in the sense that

$$r(v, x, q) = \inf_u \sup_y \{\langle v, u \rangle + xy - \tilde{r}(u, y, q)\}$$

and, for fixed q*,* (v, x) *is a saddle point for* $\tilde{r}(u, y, q)$ *if and only if* (u, y) *is a saddle point for* $r(v, x, q)$*.*

Implications of duality

- properties of r translate into properties of $\tilde{r}$
- $r \in C^2$ iff $\tilde{r} \in C^2$
- derivatives of r can be computed in terms of derivatives of $\tilde{r}$
- For conjugate saddle points (v, x) and (u, y):

$$v = \partial_u \tilde{r}(u, y, q),\ x = \partial_y \tilde{r}(u, y, q),$$

and

$$u = \partial_v r(v, x, q),\ y = \partial_x r(v, x, q)\,.$$

$\rightsquigarrow$ explicit construction of cash transfer $x = \tilde{r}(u, 1, q)$ and Pareto weights $w = \partial_u \tilde{r}(u, 1, q)/|\partial_u \tilde{r}(u, 1, q)|_1$ for given utility vector u and transaction q

An SDE for the utility process

We need to understand the martingale dynamics of our market makers' expected utilities.

Assumption

- *filtration generated by Brownian motion* B
- *contingent claims* ψ *and total initial endowment* Σ_0 *Malliavin differentiable with bounded Malliavin derivatives*
- *bounded prudence:* $\left| -\frac{u_m'''(x)}{u_m''(x)} \right| \leq K < +\infty$

Notation:

- $A(w, x, q) =$ Pareto allocation of $\Sigma_0 + (x + \langle q, \psi \rangle)$ with weights w
- $U_t(w, x, q) = \left(\mathbb{E}\left[u_m(A^m(w, x, q)) \,|\, \mathscr{F}_t \right] \right)_{m=1,\ldots,M}$
- $dU_t(w, x, q) = F_t(w, x, q)\, dB_t$

An SDE for the utility process

Theorem

For every simple strategy Q the induced process of expected utilities for our market makers solves the SDE

$$dU_t = G_t(U_t, Q_t)\, dB_t, \quad U_0 = (\mathbb{E} u_m(\alpha_0^m))$$

where

$$G_t(u, q) = F_t(W_t(u, q), X_t(u, q), q)\,.$$

Note:

This SDE makes sense for any predictable (sufficiently integrable) strategy Q!

Stability theory for SDEs

Corollary

For Q^n such that $\int_0^T (Q^n_t - Q_t)^2\,dt \to 0$ in probability, the corresponding solutions U^n converge uniformly in probability to the solution U corresponding to Q.
In particular, we have a consistent and continuous extension of our terminal wealth mapping $Q \mapsto V_T(Q)$ from simple strategies to predictable, a.s. square-integrable strategies.

Sketch of Proof:

- Use Clark-Ocone-Formula to compute F_t.
- Use assumptions on u_m and bounds on Malliavin derivatives to control dependence of G on (u, q).
- Get existence, uniqueness, stability of strong solutions to SDE.

$\square$

Conclusion

- new model for obtaining endogenous price dynamics of illiquid assets: market indifference pricing
- 2nd order expansions of transaction prices with insights into the structure of liquidity premia
- nonlinear wealth dynamics accounting for liquidity premia
- consistent and continuous extension from simple to general predictable strategies via SDE for utility process

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- 2nd order expansions of transaction prices with insights into the structure of liquidity premia
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- only a model for permanent price impact! market resilience? lack of counterparties?
- manipulable claims?

  ...

A.Ma.Me.F. workshop in Berlin

- September 27–30, 2010
- `http://sites.google.com/site/amamefberlin2010/`
- limited capacity: register soon!