Time Consistency in Portfolio Management

Traian A Pirvu

Department of Mathematics and Statistics
McMaster University

Toronto, June 2010

The talk is based on joint work with **Ivar Ekeland**

Motivation

- There are at least two examples in portfolio management that are time inconsistent:
  - Maximizing utility of intertemporal consumption and final wealth assuming a non-exponential discount rate.
  - Mean-variance utility.
- It means that the agents may have an incentive to deviate from their decisions that were optimal in the past.
- It is not often the case that management decisions are irreversible; there will usually be many opportunities to reverse a decision which, as times goes by, seems ill-advised.

Maximizing utility of intertemporal consumption and final wealth

- We work in a Black and Scholes world, with riskless rate r :

$$dS(t) = S(t)\left[\alpha\, dt + \sigma\, dW(t)\right], \quad 0 \leq t \leq \infty,$$

- Consider a self-financing portfolio. The total value is X, the amount invested in the stock is ζ and the consumption rate is c, then with $\mu = \alpha - r$:

$$dX^{\zeta,c}(t) = X^{\zeta,c}(t)[(r + \mu\zeta(t) - c(t))\, dt + \sigma\zeta(t)\, dW(t)].$$

- The investor at time $t \in [0, T]$ uses the criterion:

$$J(t, x, \zeta, c) \triangleq \mathbb{E}\left[\int_t^T h(s-t)U(c(s)X^{\zeta,c}(s))\, ds + a(T-t)U(X^{\zeta,c}(T)) \middle| X^{\zeta,c}(t) = x\right].$$

The investor's characteristics

$$J(t,x,\zeta,c) \triangleq \mathbb{E}\left[\int_t^T h(s-t)U(c(s)X^{\zeta,c}(s))\,ds + a(T-t)U(X^{\zeta,c}(T))\middle| X^{\zeta,c}(t)=x\right].$$

- T is a stopping time (death of the investor). In the sequel, we will take it as deterministic.
- $U(\cdot)$ is the utility function. In the sequel we will use $U_p(x)=\frac{x^p}{p}$ with $p<1$.
- $h(\cdot)$ is the psychological discount rate. We assume that $0 \le h(t) \le h(0)=1$, and $h(\infty)=0$.
- $a(\cdot)$ is the bequest coefficient.

Examples

$$J(t,x,\zeta,c) \triangleq \mathbb{E}\left[\int_t^T h(s-t)U(c(s)X^{\zeta,c}(s))\,ds + a(T-t)U(X^{\zeta,c}(T))\middle| X^{\zeta,c}(t)=x\right].$$

- $h(t) = a(t) = \exp(-\rho t)$. This is the classical (Merton) problem.
- $h(t) = \exp(-\rho t)$ and $a(t) \neq h(t)$.
- $h(t) = (1+at)^{-\frac{b}{a}}$ (hyperbolic discounting), $a(t) = nh(t)$.

The last two cases have strong empirical support, but they do not fall within the classical framework. Indeed, they give rise to time inconsistency on the part of the investor, so that there is no implementable optimal portfolio.

Time-inconsitency for dummies

Consider an individual who wants to stop smoking:

- If he stops today, he will suffer -1 today (withdrawal), but gain +2 tomorrow (health).
- He has a non-constant discount rate: a stream u_t is valued today $(t=0)$ at

$$u_0 + \frac{1}{2}\sum_{t=1}^{\infty} \rho^t u_t \quad \text{for some } \rho \in (\frac{1}{2}, 1)$$

- Stopping today yields a utility of $-1+\rho<0$.
- Stopping tomorrow yields a utility of $\frac{(-1+2\rho)}{2} > 0$.
- So he decides today to stop tomorrow. Unfortunately, when tomorrow comes, it becomes today, and he decides again to stop the next day.

Time-inconsistency for mathematicians

- In the exponential case, utilities discounted at time 0 and $t > 0$ are proportional

$$\mathbb{E}\left[\int_t^T e^{-\rho(s-t)} U(c(s)X^{\zeta,c}(s))\, ds + e^{-\rho(T-t)} U(X^{\zeta,c}(T))\right] =$$

$$e^{\rho t}\mathbb{E}\left[\int_t^T e^{-\rho s} U(c(s)X^{\zeta,c}(s))\, ds + e^{-\rho T} U(X^{\zeta,c}(T))\right]$$

- In the non-exponential case, this is no longer the case. The HJB equation written for the investor at time t is

$$\frac{\partial V}{\partial s}(t,s,x)+\sup_{\zeta,c}\left[(r+\mu\zeta-c)x\frac{\partial V}{\partial x}(t,s,x)+\frac{1}{2}\sigma^2\zeta^2x^2\frac{\partial^2 V}{\partial x^2}(t,s,x)\right]$$

$$+\frac{h'(s-t)}{h(s-t)}V(t,s,x)+U(xc)=0, \quad V(t,T,x)=a(T-t)U(x)$$

which obviously depends on t (so every $t-$day the investor changes his criterion of optimality).

Markov Strategies

- A Markov strategy is a pair $(F(t,x), G(t,x))$ of smooth functions.
- Investment and consumption rates are given by:

$$\zeta(t) = \frac{F(t, X(t))}{X(t)}, \qquad c(t) = \frac{G(t, X(t))}{X(t)},$$

  and the wealth then is a solution of the stochastic differential equation (SDE):

$$dX(s) = [rX(s) + \mu F(s, X(s)) - G(s, X(s))]ds + \sigma F(s, X(s))dW(s).$$

- Substituting into the criterion, we get:

$$J(t,x,F,G) \triangleq \mathbb{E}\left[\int_t^T h(s-t)U(c(s)X^{\zeta,c}(s))\,ds + a(T-t)U(X^{\zeta,c}(T))\middle| X^{\zeta,c}(t) = x\right],$$

Equilibrium Strategies

- We say that (F, G) is an equilibrium strategy if at any time t, the investor finds that he has no incentive to change it during the infinitesimal period $[t, t+\epsilon]$.

- Definition: (F, G) is an equilibrium strategy if at any time t, for every ζ and c :

$$\lim_{\epsilon \downarrow 0} \frac{J(t, x, F, G) - J(t, x, \zeta_\epsilon, c_\epsilon)}{\epsilon} \geq 0,$$

where the process $\{\zeta_\epsilon(s), c_\epsilon(s)\}_{s \in [0,T]}$ is defined by:

$$[\zeta_\epsilon(s), c_\epsilon(s)] = \begin{cases} [F(s,X(s)), G(s,X(s))] & 0 \le s \le t \\ [\zeta(s), c(s)] & t \le s \le t+\epsilon \\ [F(s,X(s)), G(s,X(s))] & t+\epsilon \le s \le T \end{cases}$$

and the equilibrium wealth process is given by the SDE:

$$dX(s) = [rX(s) + \mu F(s,X(s)) - G(s,X(s))]ds + \sigma F(s,X(s))dW(s).$$

The integral equation

- An equilibrium strategy is given by

$$F(t,x) = -\frac{\mu \frac{\partial v}{\partial x}(t,x)}{\sigma^2 \frac{\partial^2 v}{\partial x^2}(t,x)}, \;\; G(t,x) = I\left(\frac{\partial v}{\partial x}(t,x)\right), \;\; I = (U')^{-1}.$$

where v satisfies the integral equation:

$$v(t,x) = \mathbb{E}\left[\int_t^T h(s-t)U(G(s,X(s)))\,ds + a(T-t)U(X(T))\middle| X(t) = x\right],$$

where $\{X(s)\}_{s\in[0,T]}$ is given by the SDE

$$dX(s) = [rX(s) + \mu F(s,X(s)) - G(s,X(s))]ds + \sigma F(s,X(s))dW(s).$$

- In a differential form the integral equation has a non-local term.

$$\frac{\partial v}{\partial t}(t,x)+\left(rx - I\left(\frac{\partial v}{\partial x}(t,x)\right)\right)\frac{\partial v}{\partial x}(t,x)-\frac{\mu^2}{2\sigma^2}\frac{[\frac{\partial v}{\partial x}(t,x)]^2}{\frac{\partial^2 v}{\partial x^2}(t,x)}+U\left(I\left(\frac{\partial v}{\partial x}(t,x)\right)\right)=$$

$$-\mathbb{E}\left[\int_t^T h'(s-t)U\left(I\left(\frac{\partial v}{\partial x}(s,X^{t,x}(s))\right)\right)ds + a'(T-t)U(X^{t,x}(T))\right].$$

- For the special case of exponential discounting it coincides with the HJB equation since

$$\mathbb{E}\left[\int_t^T h'(s-t)U\left(I\left(\frac{\partial v}{\partial x}(s,X^{t,x}(s))\right)\right)ds + a'(T-t)U(X^{t,x}(T))\right] = -\rho v(t,x).$$

Ansatz

- Assume $U(x) = U_p(x) = \frac{x^p}{p}$. Let us look for the value function v of the form $v(t,x) = \lambda(t)x^p$, so that:

$$F(t,x) = \frac{\mu x}{\sigma^2(1-p)}, \qquad G(t,x) = [\lambda(t)]^{\frac{1}{p-1}} x.$$

- This linearizes the equilibrium wealth dynamics:

$$\frac{dX(t)}{X(t)} = \left[r + \frac{(\alpha - r)^2}{\sigma^2} \frac{1}{1-p} - [\lambda(t)]^{\frac{1}{p-1}}\right] dt + \frac{(\alpha - r)}{\sigma} \frac{1}{1-p} dW(t)$$

- The integral equation becomes

$$\begin{cases} \lambda(t) = \int_t^T h(s-t) e^{K(s-t)} [\lambda(s)]^{\frac{p}{p-1}} e^{-\left(\int_t^s p[\lambda(u)]^{\frac{1}{p-1}}\, du\right)} ds + \\ +a(T-t) e^{K(T-t)} e^{-\left(\int_t^T p[\lambda(u)]^{\frac{1}{p-1}}\, du\right)} \\ \lambda(T) = 1, \end{cases}$$

where

$$K = p\left(r + \frac{\mu^2}{2(1-p)\sigma^2}\right).$$

- This equation has a unique smooth solution.

Logarithmic utility

- Take $U(x) = \ln x$ (corresponding to $p = 0$) and $a(T-t) = nh(T-t)$. We get an explicit formula:

$$\lambda(t) = \int_t^T h(s-t)e^{K(s-t)}\, ds + nh(T-t)e^{K(T-t)},$$

and the equilibrium policies are

$$F(t,x) = \frac{\mu}{\sigma^2}x, \qquad G(t,x) = [\lambda(t)]^{-1}x.$$

Numerical Results

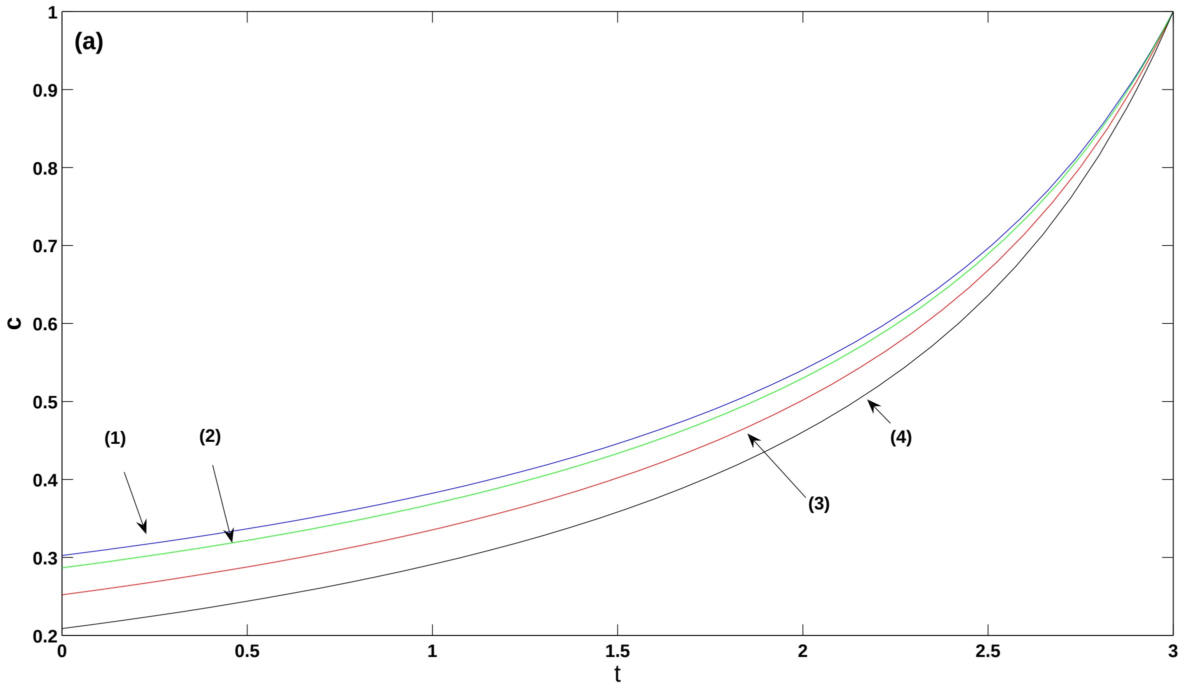
- Let us consider the following discount functions:

  exponential-$h_0(t)$, pseudo-exponential type I-$h_1(t)$ and pseudo-exponential type II-$h_2(t)$

$$h_0(t) = a_0(t) = \exp(-\rho t), \;\; h_1(t) = a_1(t) = \lambda \exp(-\rho_1 t) + (1-\lambda) \exp(-\rho_2 t),$$

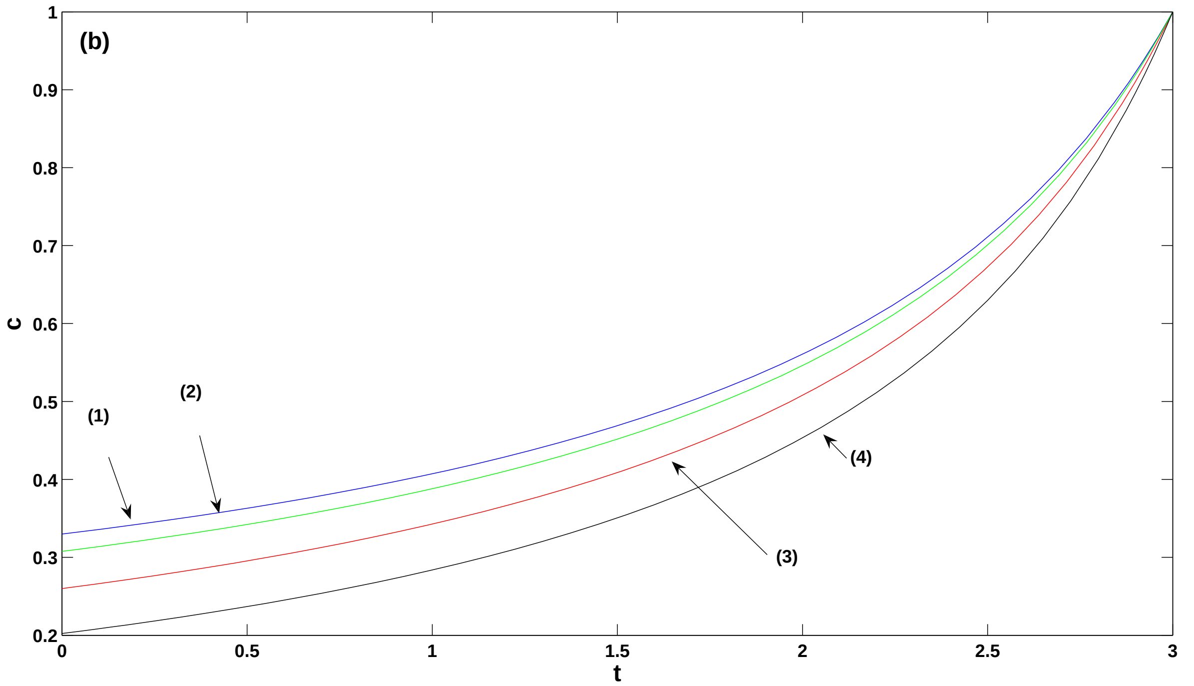
$$h_2(t) = a_2(t) = (1 + \lambda t) \exp(-\rho t).$$

- For $\alpha = 0.12$, $\sigma = 0.2$, $r = 0.05$, the discount factors $\rho_1 = 0.1$, $\rho_2 = 0.3$ and the weighting parameter $\lambda = 0.25$, CRRA $p = -1$ we graph equilibrium consumption rates



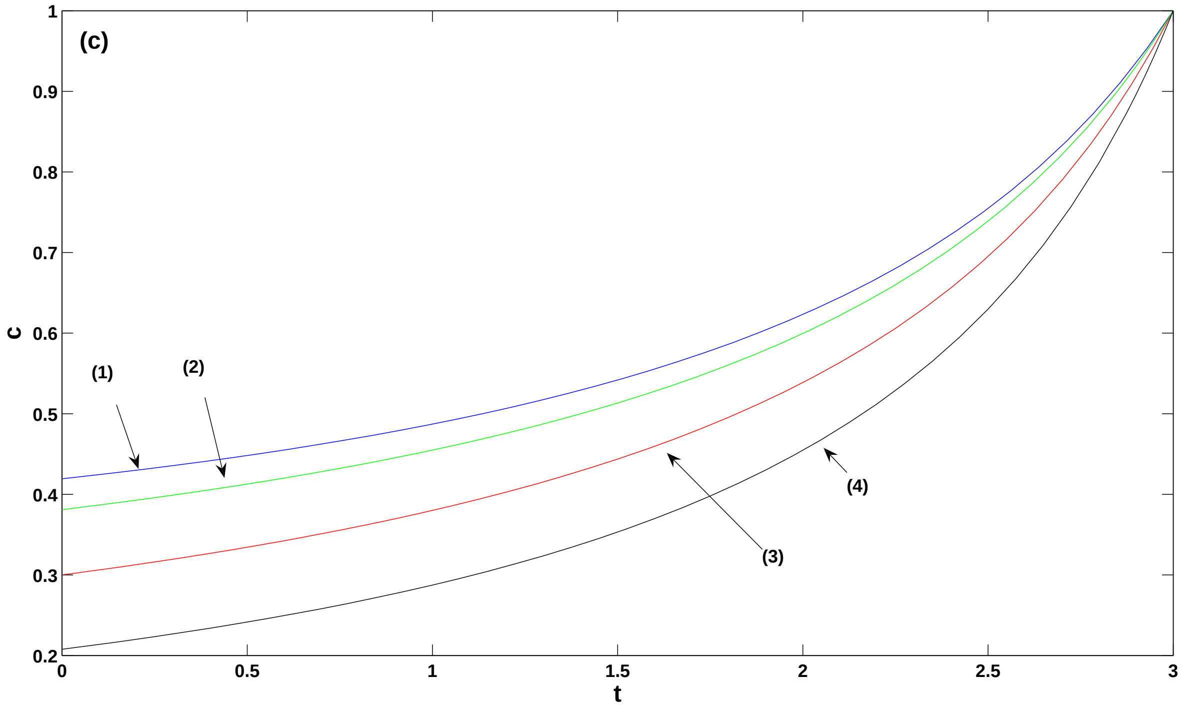
- $(1) \rightarrow \rho_2$ exp; $(2) \rightarrow \lambda, \rho_1, \rho_2$ type I ; $(3) \rightarrow \rho_1$ exp; $(4) \rightarrow \lambda, \rho_1$ type II

- Take CRRA $p = -0.5$



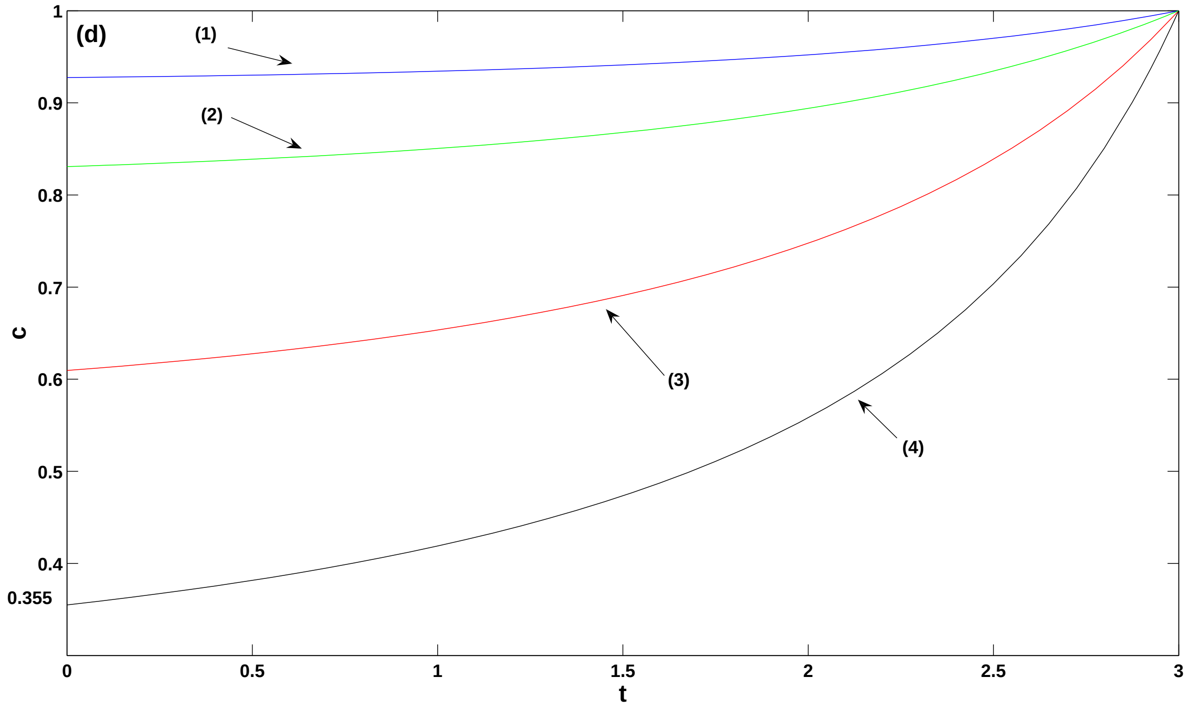
- $(1) \longrightarrow \rho_2$ exp; $(2) \longrightarrow \lambda, \rho_1, \rho_2$ type I ; $(3) \longrightarrow \rho_1$ exp; $(4) \longrightarrow \lambda, \rho_1$ type II

- Take CRRA $p = 0$



- $(1) \rightarrow \rho_2$ exp; $(2) \rightarrow \lambda, \rho_1, \rho_2$ type I ; $(3) \rightarrow \rho_1$ exp; $(4) \rightarrow \lambda, \rho_1$ type II

- Take CRRA $p = 0.5$

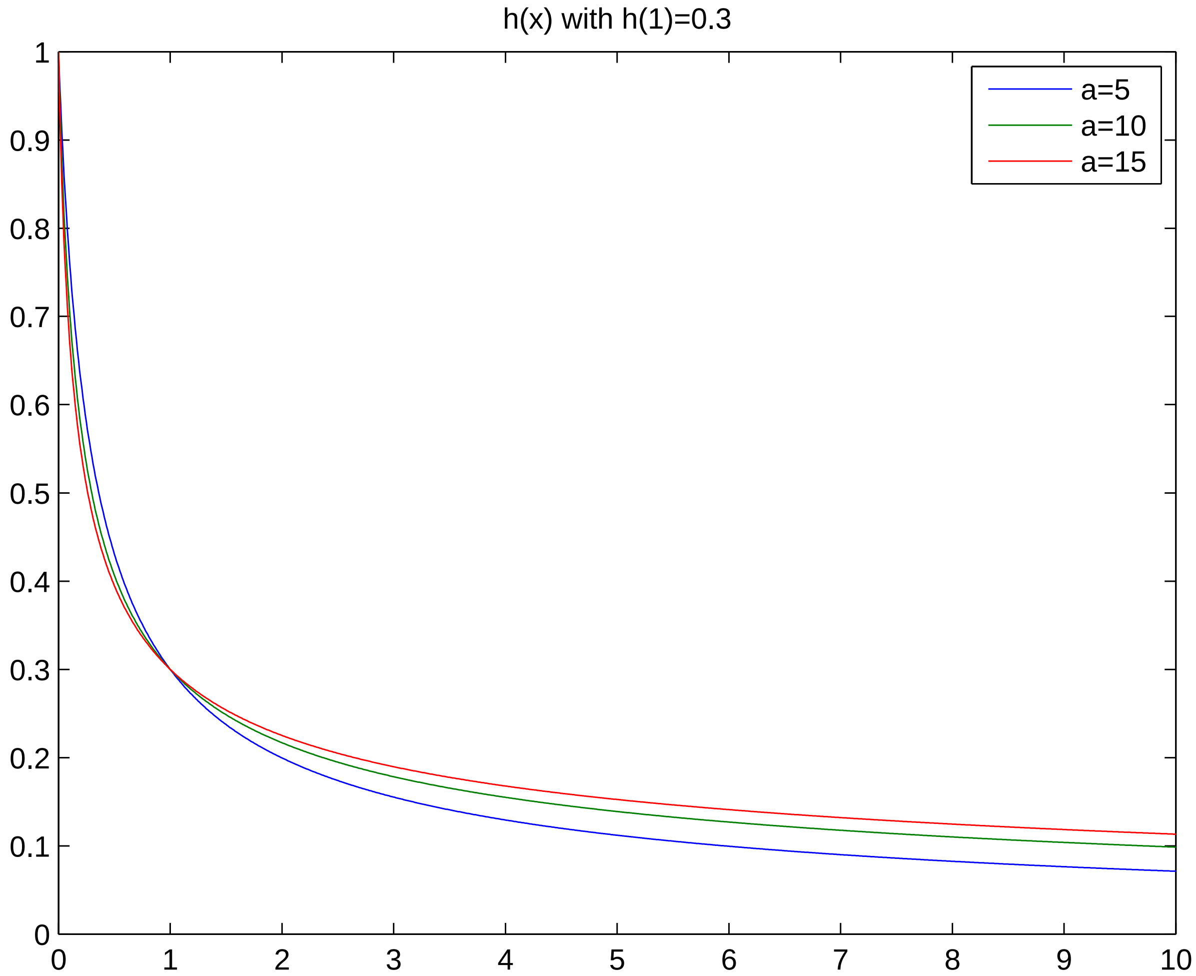


- $(1) \rightarrow \rho_2$ exp; $(2) \rightarrow \lambda, \rho_1, \rho_2$ type I ; $(3) \rightarrow \rho_1$ exp; $(4) \rightarrow \lambda, \rho_1$ type II

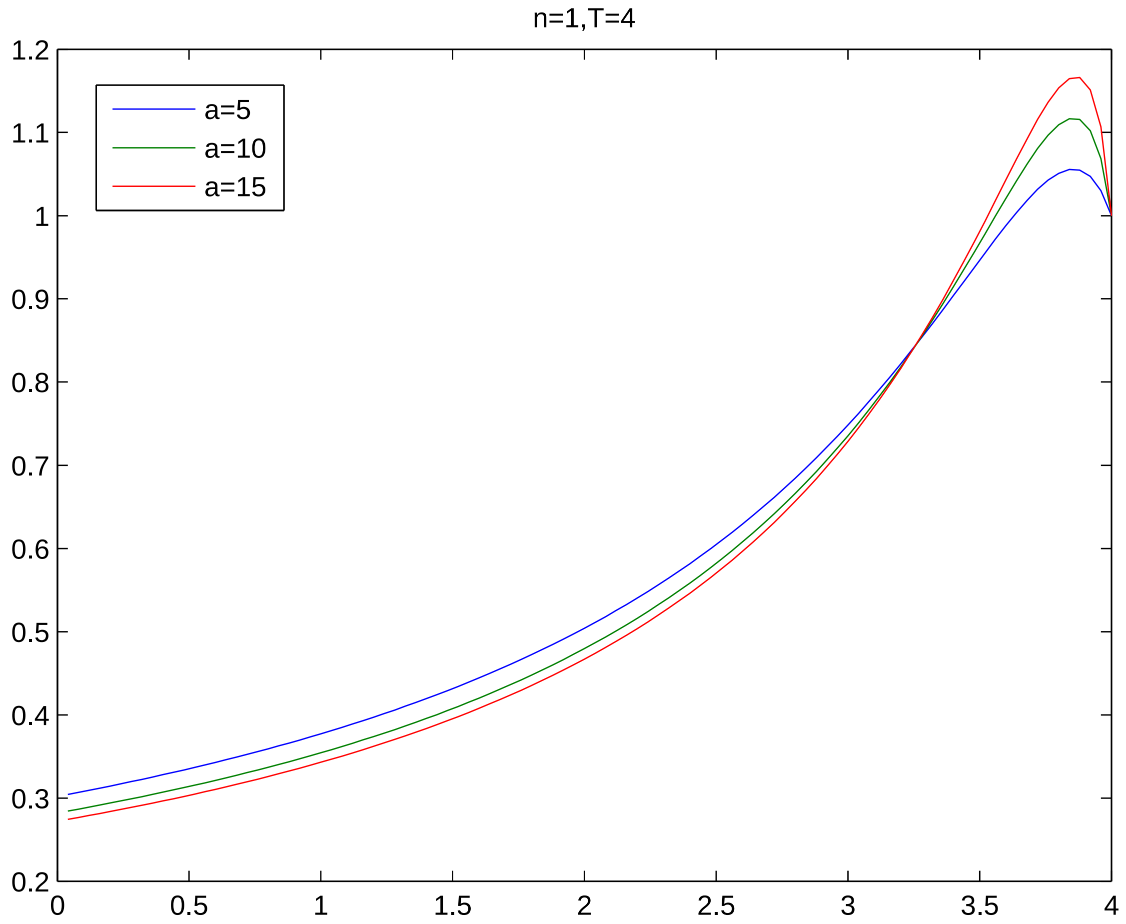
Consumption Puzzle

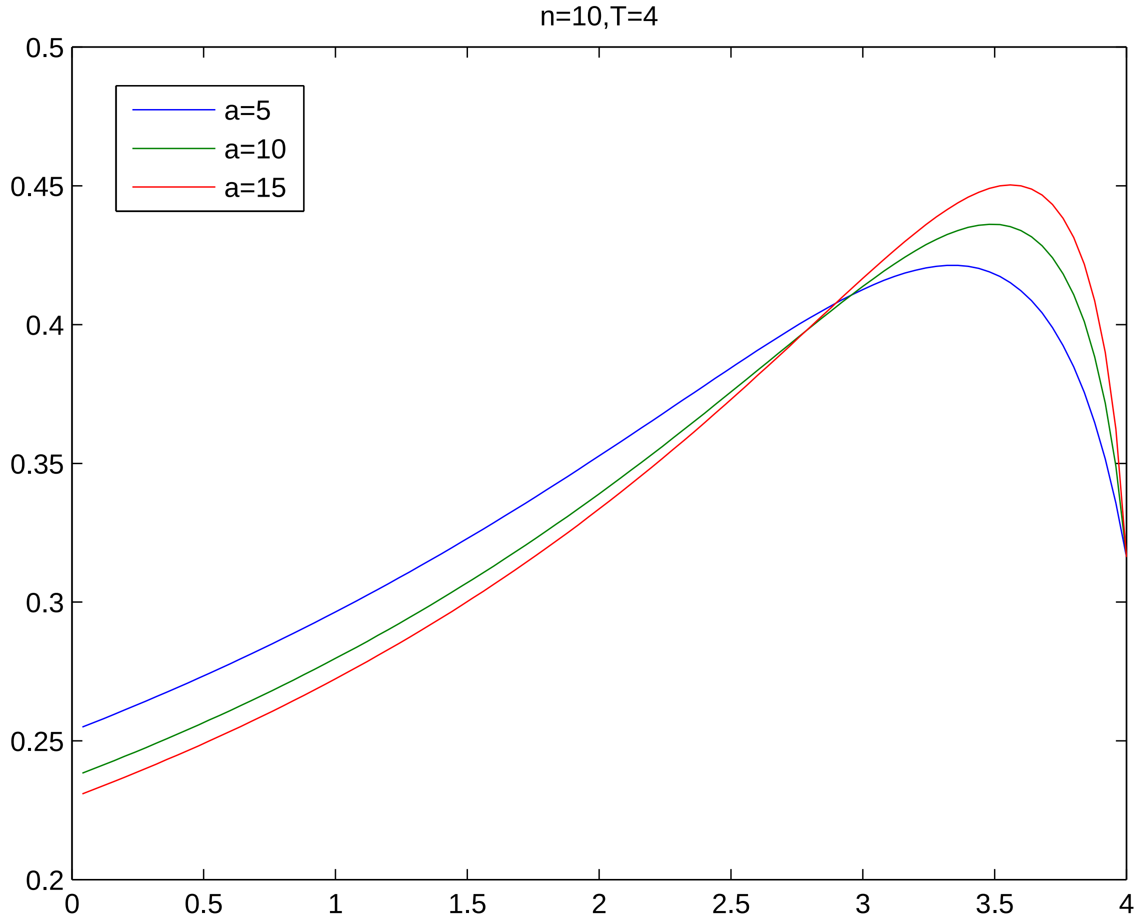
- Standard models predict that consumption will grow smoothly over time (or it will decrease smoothly).
- Household data indicate that consumption is hump-shaped.
- This inconsistency is known as the consumption puzzle.
- Hyperbolic discounting can explain this puzzle. Let us consider:

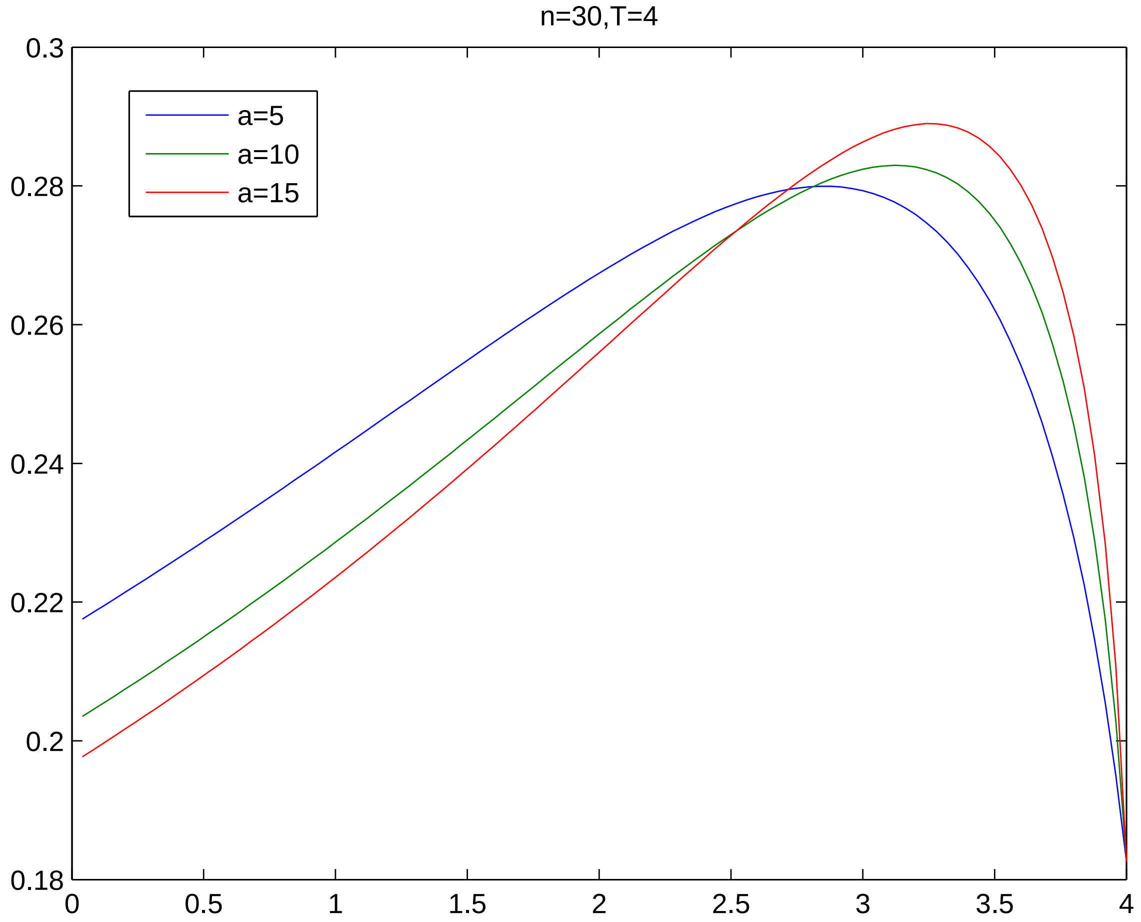
$$h(t) = (1 + at)^{-\frac{b}{a}}, \qquad a(t) = n(1 + at)^{-\frac{b}{a}}.$$


h(x) with h(1)=0.3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
9
10
a=5
a=10
a=15

- Let us graph equilibrium consumption rates




n=10,T=4
a=5
a=10
a=15
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0
0.5
1
1.5
2
2.5
3
3.5
4


n=30,T=4
a=5
a=10
a=15
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0
0.5
1
1.5
2
2.5
3
3.5
4

Mean Variance Utility

- The risk criterion is:

$$J(t,x,\zeta) = \mathbb{E}[X^{\zeta}(T)|X^{\zeta}(t)=x] - \frac{\gamma}{2}Var(X^{\zeta}(T)|X^{\zeta}(t)=x).$$

- Time inconsistency arrises from the risk criterion non-linearity wrt expected value of the terminal wealth.

- Bjork and Murgoci found that the equilibrium investment is

$$F(t,x) = \frac{1}{\gamma}\frac{\mu}{\sigma^2}e^{-r(T-t)}.$$

The End!

Thank You!