Exponential Hedging under Variable Horizons

Tahir Choulli

Math & Stats Sciences Dept., University of Alberta, Edmonton

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This talk is based on joint work with **Junfeng Ma (U of A) and Marie-Amelie Morlais (Lemans)**

Main points of the talk

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- **Answers(partial) to the main problems**
- **Main Mathematical/Statistical Tool: Minimal Hellinger Local Martingale Densities**

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- Economics Literatures: Yaari, M.E. (1965), Hakansson, N.H. (1969), Richard, S. (1975), Fischer(1930),..., etcetera.
- Mathematical Economics/Finance, Continuous models, Investment/hedging problems? Choulli and Schweizer (2009).

- Our approach: Consider a "nice" random time τ,

$$\begin{aligned} EU(x+G(\theta)_\tau - H_\tau) &= E\int_0^{+\infty} U(x+G(\theta)_t - H_t)dF_t \\ &= \int_0^{\infty} EU(x+G(\theta)_{\tau_t} - H_{\tau_t})dt. \end{aligned}$$

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- How far is problem is close to the self-generating utilities (the forward utilities introduced by Musiela and Zariphoupoulou)...

The market model

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- $$\mathcal{M}^e_f(S) := \{Q \mid Q \sim P, \quad S \text{ is a } Q-\text{local martingale}, \\ H(Q|P) = E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right] < +\infty\} \neq \emptyset$$

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- Consider a semimartingale B such that there exists $p > 1$,

$$\sup_{\tau\in\mathcal{T}} E(e^{pB_\tau}) < +\infty. \tag{1}$$

- Suppose that the dynamic claim B is bounded from below and satisfies (1), then we denote

$$\Theta(S, B) = \text{the set of admissible strategies} \tag{2}$$

  Similar admissibility as in Delbean et al. (2002) or Kabanov and Stricker (2002), but for any stopping time.

See the works of Musiela/Zariphoupoulou since early 2003, the paper of Hobson/Henderson (2007), Choulli/Stricker (2005, 2006), and Choulli/Stricker/Li(2007).

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  Similar admissibility as in Delbean et al. (2002) or Kabanov and Stricker (2002), but for any stopping time.
- A random field utility $U(t, \omega, x) = U(t, x)$ is a forward utility if there exists "*an admissible strategy*" $\widetilde{\theta}$ such that $U(t, x + (\widetilde{\theta} \cdot S)_t)$ is a true martingale, and for any admissible strategy θ, $U(t, x + (\theta \cdot S)_t)$ is a supermartingale.

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Main Problems

- **Exponential Forward utilities** Explicit description of semimartingales B such that the following utility $U(t,\omega,x) = -exp\left(-x + B_t(\omega)\right)$, is a forward utility.

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- **Horizon-unbiased exponential hedging** Explicit description of semimartingale B for which we can find $\widehat{\theta} \in \Theta(S,B)$ such that for any $\tau \in \mathcal{T}_b$,

$$\max_{\theta\in\Theta(S,B)} E\left\{-exp\left(B_\tau - (\theta\cdot S)_\tau\right)\right\} = E\left\{-exp\left(B_\tau - (\widehat{\theta}\cdot S)_\tau\right)\right\}. \tag{3}$$

Explicit description of the strategy $\widehat{\theta}$.

- **Horizon-dependence structure in optimal strategy/wealth** Given bounded semimartingale, B, for which there exists a two-parameters process $\left(\widehat{\theta}(t,s)\right)_{0\leq s,t\leq T}$, such that for ant $t \in (0,T]$, $\theta(t,\cdot)$ is solution

$$\max_{\theta\in\Theta(S,B)} E - exp\left[B_t - G_t(\theta)\right].$$

Determine the structures of the optimal wealth over horizon, i.e., $\widehat{X}_t := \int_0^t \widehat{\theta}(t,u)dS_u, \quad 0 \leq t \leq T.$

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- **Optimal Sale Problem** Given B find/describe -as explicit as possible- $\widetilde{\theta}$ and τ^* s.t.

$$\inf_{\tau\in\mathcal{T}} \inf_{\theta\in\Theta(S,B)} Eexp\left[B_\tau - G_\tau(\theta)\right] = Eexp\left[B_{\tau^*} - G_{\tau^*}(\widetilde{\theta})\right].$$

Horizon-unbiased vs Forward utility

Proposition: The following are equivalent:

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- The following utility $U(t,\omega,x) = -exp\left(-x + B_t(\omega)\right)$, is a forward utility.
- The horizon-unbiased problem in (3) admits a solution and the optimal value function, $u(\tau)$, does not depend on the horizon τ. That is $u(\tau) = u(0)$, $\forall\ \tau \in \mathcal{T}$.

Explicit description of Exponential Forward utilities

The predictable case

Theorem 1: Consider a bounded predictable process with finite variation denoted by B. Then, the following assertions are equivalent:

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- (i) The random utility

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  is a forward utility.
- (ii) The Minimal entropy-Hellinger martingale measure, $\widetilde{Q}$, exists and

  $$B = B_0 + h^E(\widetilde{Q}, P).$$

- Through out the remaining part of this section, we suppose given a semimartingale, B, that is bounded from below and satisfying (1). Consider the Multiplicative Doob-Meyer decomposition
$$e^B = \mathcal{E}(M^{(B)})e^{B'}. \tag{4}$$
$M^{(B)}$ is a local martingale and B' is a predictable process with finite variation and $B'_0 = B_0$.
- For a numéraire, N, given by $N := \mathcal{E}(\pi \cdot S)$ ($\pi \in L(S)$), we associate the new discounted asset prices process
$$\overline{S} := S - \frac{\pi}{1 + \pi^* \Delta S} \cdot [S, S]. \tag{5}$$

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$$B - B_0 = \log\left[\mathcal{E}(M^{(B)})\right] + h^E\left(\widetilde{Z}^{(B)}|\mathcal{E}(M^{(B)})\right). \qquad (6)$$

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  - (b) The process $\widehat{Z}^{(B)} := \mathcal{E}\left(M^{(B)}\right)\widetilde{Z}^{(B)}$ represents the density of a martingale measure, $\widehat{Q}^{(B)}$, that has finite P-entropy, i.e.,$H\left(\widehat{Q}^{(B)}|P\right) := E\left\{\widehat{Z}^{(B)}_T \log\left[\widehat{Z}^{(B)}_T\right]\right\} < +\infty$,

The problem

Consider a bounded from below semimartingale, B satisfying (1) and a numéraire, N, that we suppose bounded away from zero and from above (to simplify). We are interested in addressing our four problems for the utility function:

$$U(t, \omega, x) = -exp\left[-\frac{x - B_t(\omega)}{N_t(\omega)}\right].$$

Transformation for Horizon-Unbiased Problem

Theorem 3: Then following assertions are equivalent:

- (i) There exists a strategy $\widetilde{\theta} \in \Theta(S, B)$ s.t. $\forall\, \tau \in \mathcal{T}_b$,

$$\min_{\theta \in \Theta(S,B)} Eexp\left[\frac{1}{N_\tau}\Big(B_\tau - (\theta \cdot S)_\tau\Big)\right] = Eexp\left[\frac{1}{N_\tau}\Big(B_\tau - (\widetilde{\theta} \cdot S)_\tau\Big)\right]. \tag{7}$$

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- (ii) There exists a strategy $\widetilde{\varphi} \in \Theta(\overline{S}, \frac{B}{N})$ s.t. $\forall\ \tau \in \mathcal{T}_b$,

$$\min_{\varphi \in \Theta(\overline{S}, \frac{B}{N})} Eexp\left[\frac{B_\tau}{N_\tau} - (\varphi \cdot \overline{S})_\tau\right] = Eexp\left[\frac{B_\tau}{N_\tau} - (\widetilde{\varphi} \cdot \overline{S})_\tau\right]. \tag{8}$$

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- Furthermore, $\widetilde{\theta}$ and $\widetilde{\varphi}$ satisfy $\widetilde{\varphi} = \Big(\widetilde{\theta} \cdot S\Big)_{-} \pi - \widetilde{\theta}$.

Theorem 4:
The random utility functional,

$$U(t,\omega,x) = -exp\left(-\frac{x - B_t(\omega)}{N_t(\omega)}\right),$$

is a forward utility performance for S if and only if the random utility

$$\overline{U}(t,\omega,x) = -exp\left[-x + \frac{B_t(\omega)}{N_t(\omega)}\right],$$

is a forward utility performance for $\overline{S}$ defined in (5).

Horizon-unbiased exponential hedging

Predictable Dynamic Claim

Theorem 5:

- Let B be a bounded predictable process with finite variation, for which there exists $\widehat{\theta} \in \Theta(S, B)$ and for any $\tau \in \mathcal{T}$, (3) holds. That is

$$\max_{\theta \in \Theta(S,B)} -E\left[exp\Big(B_\tau - (\theta \cdot S)_\tau\Big)\right] = -E\left[exp\Big(B_\tau - (\widehat{\theta} \cdot S)_\tau\Big)\right]. \tag{9}$$

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Then the following hold:

- (i) $\widehat{\theta}$ coincides with $-\widetilde{\theta}^{MEH}$ defined in (12).
- (ii) The processes $B$and $h^E(\widetilde{Z}, P)$ coincide on

$$\{(t,\omega) | \text{ there exists } \theta \in \Theta(S,B) \quad \text{such that} \quad (\theta \cdot S)_{t-}(\omega) \neq 0\}$$

Theorem 6: Consider a bounded from below semimartingale, B, satisfying (1), and consider the processes $M^{(B)}$ and B' given in (4). Suppose that there exists $\widehat{\theta} \in \Theta(S, B)$ such that for any $\tau \in \mathcal{T}_b$,

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Then, the following assertions are fulfilled:

- (a) The process $\widehat{\theta}$ coincides with $-\widetilde{\theta}^{MEH}(B)$ given by the decomposition of the MEHM local martingale density, $\widetilde{Z}^{(B)}$, with respect to $\mathcal{E}\left(M^{(B)}\right)$ via the equation (12).

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- (b) The two processes B' and $h^E(\widetilde{Z}^{(B)}|M^{(B)})$ coincide on

$$\{(t,\omega)\ |\ \text{there exists } \theta \in \Theta(S,B)\ \text{ such that }\ (\theta\cdot S)_{t-}(\omega) \neq 0\}.$$

Theorem 7: For any $t \in (0, T]$, $\widehat{X}_t := \int_0^t \widehat{\theta}(t,u)dS_u$ is the optimal wealth in exponentially hedging the claim B_t. Then the following assertions hold:

- (i) The process $\widehat{X}$ has a càdlàg modification that is a semimartingale.

Choulli and Schweizer (2009).

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- (ii) The two-parameters process $\left(\widehat{\theta}(t,s)\right)_{0\leq s,t\leq T}$ has version that has finite variation in the first parameter t, and its diagonal $\widehat{\theta}_t := \widehat{\theta}(t,t)$ is S-integrable.

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- (iii) The process $\widehat{X} - (\widetilde{\theta} \cdot S)$ is predictable with finite variation.

Choulli and Schweizer (2009).

Optimal Sale problem

Main Results

- The process
$$V(t) := ess \sup_{\tau\geq t,\ \theta\in\Theta} E\left(-exp\Big[-\int_t^\tau \theta_u dS_u + B_\tau\Big] |\mathcal{F}_t\right).$$
admits a càdlàg modification and satisfies
$$V(t) = \max\left[-e^{B_t}; ess\sup_{\tau>t,\ \theta\in\Theta} E\left(V_\tau e^{-\int_t^\tau \theta_u dS_u} |\mathcal{F}_t\right)\right].$$

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- V is a càdlàg negative semimartingale s.t.

  $$V(t) = V(0)\mathcal{E}(M^V)e^{A^V},\ \ M^V = \beta\cdot S^c + (f-1)\star(\mu-\nu) + g\star\mu + M'.$$

  Here A^V is a predictable process with finite variation, M^V is a local martingale, and (β, f, g, M') are the Jacod's components for M^V.

The Optimal Time

The optimal investment time is given by

$$\tau^* = \tau_0 := \inf\{0 \leq t \leq T \mid V(t) = -e^{B_t} \ \ or \ \ V(t-) = -e^{B_{t-}}\} \wedge T.$$

That is $V(0) = \sup\limits_{\theta \in \Theta} E\left[-e^{-(\theta \cdot S)_{\tau^*} + B_{\tau^*}}\right]$, or more generally

$$V(t) = ess\,\sup_{\theta \in \Theta} E\left(-exp\left[-\int_t^{\tau_t} \theta_u dS_u + B_{\tau_t}\right] | \mathcal{F}_t\right),$$
$$\tau_t := \inf\{u \in [t, T] \mid V(u) = -e^{B_u} \ \ or \ \ V(t-) = -e^{B_{t-}}\} \wedge T.$$

The Optimal Strategy

If the optimal strategy θ^* exists, then on $[0, \tau^*]$ it is the pointwise root of

$$0 = b + c(\theta - \beta) + \int \left[1 - f(x)e^{-\theta^T x}\right] xF(dx).$$

Particular Cases

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- **Proposition 1:** Consider a semimartingale, B satisfying assumptions of Theorem 6. Then the following assertions hold:
  (i) If $h^E\left(\widetilde{Z}^{(B)}, \mathcal{E}(M^{(B)})\right) \preceq B'$, then the optimal investment time τ^* coincides with the initial time 0. That is the best option to invest is to liquidate the real asset immediately.

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- (ii) If $B' \preceq h^E\left(\widetilde{Z}^{(B)}, \mathcal{E}(M^{(B)})\right)$, then the optimal investment time τ^* coincides with the horizon T. That is the best option to invest is to wait to the maturity to liquidate the real asset.

Proposition 2: Suppose that $B' - h^E\left(\widetilde{Z}^{(B)}, \mathcal{E}(M^{(B)})\right)$ increases on (τ_{2i}, τ_{2i+1}) and decreases on $(\tau_{2i+1}, \tau_{2i+2})$, $i = 0, ..., N$, then the quantity $\inf_{\tau\in\mathcal{T}} \inf_{\theta\Theta(S,B)} Ee^{B_\tau - G_\tau(\theta)}$ coincides with

$$\min\{\inf_{\theta\in\Theta(S,B)} Eexp\left[B_{T_{2i+1}} - G_{T_{2i+1}}(\theta)\right],\ i = 0, 1, ..., N\}.$$

Minimal Hellinger martingale densities

- Let $N \in \mathcal{M}_{0,\ loc}(P)$ such that $1 + \Delta N \geq 0$. If
$$V_t^{(E)}(N) := \frac{1}{2}\langle N^c \rangle_t + \sum_{0<s\leq t} \left[(1+\Delta N_s)\log(1+\Delta N_s) - \Delta N_s\right] \tag{11}$$
is locally integrable, then its compensator (with respect to the probability P) is called the entropy- Hellinger process of N, and is denoted by $H^{(E)}(N,\ P)$.
- Let $Q \in I\!P_a$ with density $Z = \mathcal{E}(N)$. If $V_t^{(E)}(N)$ is locally integrable, then its compensator is called the entropy-Hellinger process of Q with respect to P, and we denoted by
$$h_t^E(Q,\ P) := h_t^E(Z,\ P) := H_t^E(N,\ P), \quad 0 \leq t \leq T.$$

- We call minimal entropy-Hellinger martingale measure, $\widetilde{Q}$, is the martingale measure -when it exists- solution to

$$\min_{Q \in \mathcal{M}^e_f(S)} h^E(Q, P),$$

with respect to the following order: $X \preceq Y$ if $Y - X$ is nondecreasing.

- When minimizing the densities over the set of local martingale densities Z that are $Zlog(Z)$ locally integrable, we obtain the minimal entropy-Hellinger local martingale density, that we denote $\widetilde{Z}$. It is explicitly calculated and is given by

$$\log\left[\widetilde{Z}\right] = \widetilde{\theta}^{MEH} \cdot S + h^E(\widetilde{Z}, P). \tag{12}$$

- The predictable processes $\widetilde{\theta}^{MEH}$ and $h^E(\widetilde{Z}, P)$ are explicitly

- Let Q be a probability measure and Y is a Q-local martingale such that $1+\Delta Y \geq 0$. Then if the càdlàg nondecreasing process

$$V^E(Y) = \frac{1}{2}\langle Y^c\rangle + \sum \left[(1+\Delta Y)\log(1+\Delta Y) - \Delta Y\right], \quad (13)$$

is Q-locally integrable (i.e. $V^E(Y) \in \mathcal{A}^+_{loc}(Q)$), then its Q-compensator is called the entropy-Hellinger process of Y (or equivalently of $\mathcal{E}(Y)$) with respect to Q, and is denoted by $h^E(Y,Q)$ (respectively $h^E(\mathcal{E}(Y),Q)$).

- Let $N,\ \overline{N} \in \mathcal{M}_{0,\ loc}(P)$ such that $1+\Delta N \geq 0$ and $1+\Delta\overline{N} \geq 0$. If the càdlàg nondecreasing process

$$\frac{1}{2}\langle N^c - \overline{N}^c\rangle + \sum \left\{ (1+\Delta N)\log\left[\frac{1+\Delta N}{1+\Delta\overline{N}}\right] - \Delta N + \Delta\overline{N} \right\}, \tag{14}$$

is locally integrable, then its compensator is called the entropy-Hellinger process of $\mathcal{E}(N)$ with respect to $\mathcal{E}(\overline{N})$, and is denoted by $h^E\left(\mathcal{E}(N)|\mathcal{E}(\overline{N})\right)$.

- If $Q_n := \mathcal{E}_{T_n}(\overline{N}) \cdot P$, then

$$h^E_{t\wedge T_n}\left(\mathcal{E}(N)|\mathcal{E}(\overline{N})\right) = h^E_{t\wedge T_n}\left(\mathcal{E}(N), Q_n\right).$$

Principal Works

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Thank you for your attention