Numerical Inverse Helmholtz Problems with Interior Data

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Supported by the National Science Foundation

Fields-MITACS Conference on Mathematics of Medical Imaging

June 20, 2011

Presentation Outline

1. Motivation by TAT
2. Reconstruction of Refractive Index
3. Reconstruction of Conductivity
4. Reconstruction of Both Coefficients
5. Concluding Remarks

Presentation Outline

Thermoacoustic Tomography (TAT): Setup

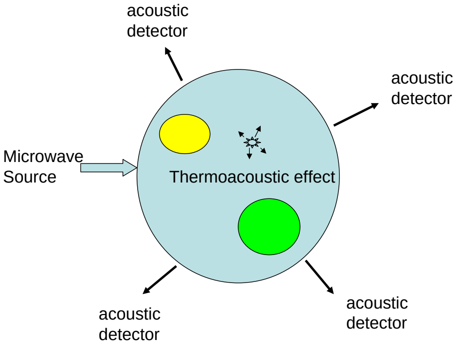

Figure 1: Thermoacoustic Tomography (TAT): To recover conductivity coefficient and refractive index properties of tissues from boundary measurement of acoustic signal. Two well-separated wave propgation processes: microwave radiation and acoustic radiation.

Thermoacoustic Tomography: Mathematical Models

- **Microwave Radiation (in Scalar Case):**

$$\begin{aligned} \Delta u(\mathbf{x}) + k^2(1+n(x))u(\mathbf{x}) + ik\sigma(x)u(x) &= 0, && \text{in } X \\ u &= g(\mathbf{x}), && \text{on } \partial X \end{aligned}$$

- **Acoustic Radiation:**

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - c^2(\mathbf{x})\Delta p &= 0, && \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ p(0,\mathbf{x}) &= H \equiv \sigma(\mathbf{x})|u|^2(\mathbf{x}), && \text{in } \mathbb{R}^d \\ \frac{\partial p}{\partial t}(0,\mathbf{x}) &= 0, && \text{in } \mathbb{R}^d \end{aligned}$$

- **Available Acoustic Data:** $p(t,\mathbf{x})$ on $\mathbb{R}_+ \times \partial X$
- **Objective:** To reconstruct $\sigma(\mathbf{x})$ and $n(\mathbf{x})$ from measured data.

TAT: Acoustic Reconstructions

- When c is constant, the Poisson-Kirchhoff formula gives

$$p(t,\mathbf{x}) = c\frac{\partial}{\partial t}\Big(t\big(\int_{|\mathbf{y}|=1} H(\mathbf{x}+t\mathbf{y})d\mu(\mathbf{y})\big)\Big)$$

- Acoustic reconstruction: The sphere mean operator can be inverted stably to recover $H(\mathbf{x})$ with full (and sometimes partial) data: Agranovsky, Ambartsoumian, Ammari, Arridge, Finch, Haltmeier, Kuchment, Kunyansky, Nguyen, Patch, Quinto, Scherzer, Wang, and many more.
- When c is not constant, acoustic inversion is much more complicated: Hristova-Kuchment-Nguyen 08, Hristova 09, Stefanov-Uhlmann 09.
- In this talk, we assume that H is reconstructed alreday.

TAT: Qualitative vs. Quantitative

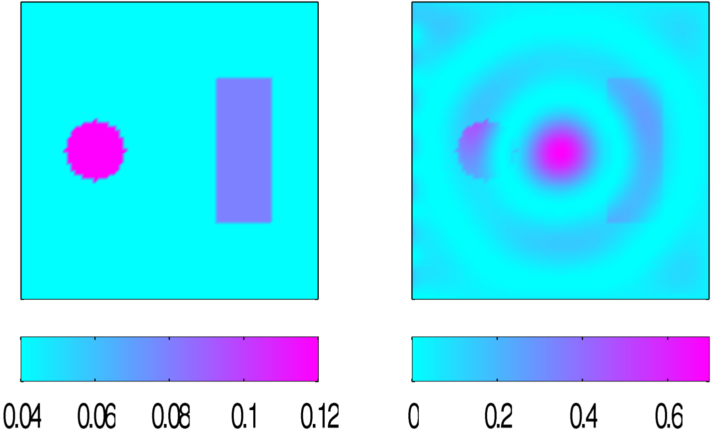

Figure 2: Necessarity of qTAT. Left: True conductivity coefficient $\sigma(\mathbf{x})$; Right: $H(\mathbf{x}) = \sigma(\mathbf{x})|u|^2(\mathbf{x})$.

Presentation Outline

The Quantitative TAT Problem

- Mathematical model is the Helmholtz equation:

$$\begin{aligned} \Delta u(\mathbf{x}) + k^2(1+n(x))u + ik\sigma(\mathbf{x})u(\mathbf{x}) &= 0, \quad \text{in } X \\ u &= g(\mathbf{x}), \quad \text{on } \partial X \end{aligned}$$

- Interior data is the result from acoustic inversion:

$$H(\mathbf{x}) = \sigma(\mathbf{x})|u|^2(\mathbf{x})$$

- Objective: to reconstruct $\sigma(\mathbf{x})$ and $n(\mathbf{x})$ from interior data H.
- In general, more than one data sets are needed to reconstruct two coefficients simultaneously.
- Main assumptions: ∂X is smooth, $\Delta + k^2(1+n(x))$, $\Delta + ik\sigma(x)$ and $\Delta + k^2(1+n(x)) + ik\sigma(x)$ with the boundary conditions are invertible.

The Quantitative TAT Problem

Let us separate the amplitude and phase by introducing the notation $u = |u|e^{i\varphi}$. Then the Helmholtz equation becomes

$$\begin{aligned} \Delta|u| + k^2(1+n)|u| &= |u||\nabla\varphi|^2, && \text{in } X \\ \nabla\cdot|u|^2\nabla\varphi + k\sigma|u|^2 &= 0, && \text{in } X \\ |u| = |g| \quad &, \quad \varphi = \varphi_g && \text{on } \partial X \end{aligned}$$

Lemma

Let H and $\tilde{H}$ be two data sets corresponding to (σ, n) and $(\tilde{\sigma}, \tilde{n})$. Then $H = \tilde{H}$ implies

$$\sqrt{\frac{\sigma\tilde{\sigma}}{H^2}}\nabla\cdot\frac{H}{\sigma}\nabla\sqrt{\frac{\tilde{\sigma}}{\sigma}} + k(n-\tilde{n}) = |\nabla\varphi|^2 - |\nabla\tilde{\varphi}|^2, \qquad \text{in } X$$

$$\nabla\cdot H\Big(\frac{\nabla\varphi}{\sigma} - \frac{\nabla\tilde{\varphi}}{\tilde{\sigma}}\Big) = 0, \qquad \text{in } X$$

Unique Recovery of Refraction Index

- Uniqueness in Recovery of $n(x)$. If $\sigma = \tilde{\sigma}$, then the second equation in the Lemma plus identical boundary data $\varphi_g = \varphi_{\tilde{g}}$ imply that $\varphi = \tilde{\varphi}$ in X. Then the first equation in the Lemma implies $n(x) = \tilde{n}(x)$ in X.
- The Reconstruction Strategy. The reconstruction can be done as follows
  - Solve the elliptic equation

$$\begin{aligned} \nabla \cdot \frac{H}{\sigma} \nabla \varphi + kH &= 0, \quad \text{in } X \\ \varphi &= \varphi_g \quad \text{on } \partial X \end{aligned}$$

  - Reconstructing $1 + n(x)$ as

$$1 + n(x) = \frac{|u||\nabla \varphi|^2 - \Delta |u|}{k^2 |u|}$$

Unique Recovery of Refraction Index: Simulation

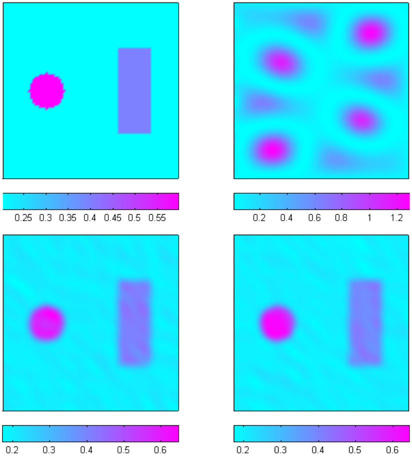

Figure 3: Top left to bottom right: real n, data $H(x)$, reconstruction with minimization, and reconstruction with direct method.

Presentation Outline

Recovery of Conductivity

- The Recovery of $\sigma(x)$. If $n(x)$ is known, then σ can be uniquely reconstructed if the following nonlinear system admits a unique solution

$$\begin{array}{rcll} \Delta|u| + k^2(1+n)|u| & = & |u||\nabla\varphi|^2, & \text{in } X \\ \nabla\cdot|u|^2\nabla\varphi + kH & = & 0, & \text{in } X \\ |u| = |g| & , & \varphi = \varphi_g & \text{on } \partial X \end{array}$$

- The Reconstruction Strategy. The following fixed point iteration works very well numerically.
  - $j = 0$, guess σ_0, construct $|u|_0 = \sqrt{H/\sigma_0}$
  - $j \geq 1$

$$\begin{array}{rcll} \Delta|u|_j + k^2(1+n)|u|_j & = & |u|_j|\nabla\varphi_j|^2, & \text{in } X \\ \nabla\cdot|u|_{j-1}^2\nabla\varphi_j + kH & = & 0, & \text{in } X \\ |u|_j = |g| & , & \varphi_j = \varphi_g & \text{on } \partial X \end{array}$$

  - reconstruct σ as $\sigma = H/|u|_\infty^2$.

Unique Recovery of Conductivity

The following result on the unique determination of the conductivity has been proven using a different strategy.

Theorem (Bal-R.-Zhou-Uhlmann, IP 11)

Let $\sigma, \tilde{\sigma} \in \mathcal{M} \equiv \{f \in \mathcal{H}^p(X) | \, \|f\|_{\mathcal{H}^p(X)} \leq M\}$ *be two conductivity coefficients. There exists an illumination* $g \in \mathcal{H}^{p-\frac{1}{2}}(\partial X)$ *such that*

$$H = \tilde{H} \implies \sigma = \tilde{\sigma}.$$

Moreover,

$$\|\sigma - \tilde{\sigma}\|_{\mathcal{H}^p(X)} \leq C\|H - \tilde{H}\|_{\mathcal{H}^p(X)},$$

with C *is independent of* σ *and* $\tilde{\sigma}$ *in* $\mathcal{M}$.

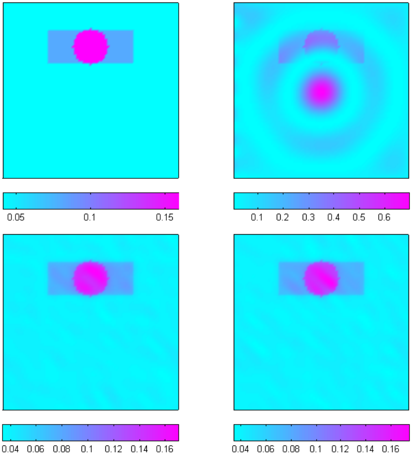

Figure 4: Top left to bottom right: real σ, data H, σ reconstructed with fixed point iteration and the method in Bal-R.-Zhou-Uhlmann, IP 11.

Presentation Outline

Quantitative TAT Problem

- Let us assume we have N_S illuminations g_s, $1 \leq s \leq N_S$. The scalar Helmholtz equation:

$$\begin{aligned} \Delta u_s(\mathbf{x}) + k_s^2(1+n(x))u_s + ik_s\sigma(\mathbf{x})u_s(\mathbf{x}) &= 0, \quad \text{in } X \\ u_s &= g_s(\mathbf{x}), \quad \text{on } \partial X \end{aligned}$$

- Interior data is the result from acoustic inversion:

$$H_s(\mathbf{x}) = \sigma(\mathbf{x})|u_s|^2(\mathbf{x}), \qquad 1 \leq s \leq N_S$$

- Objective: to reconstruct $\sigma(\mathbf{x})$ and $n(\mathbf{x})$ simultaneously.
- Main assumptions: ∂X is smooth, $\Delta + k_s^2(1+n(x))$, $\Delta + ik_s\sigma(x)$ and $\Delta + k_s^2(1+n(x)) + ik_s\sigma(x)$ with the boundary conditions are invertible.

Reconstruction Strategy I: Numerical Optimization

- Least-square formulation. We formulate the inverse problem as the following nonlinear least-square problem. That is, to minimize the following mismatch functional

$$\mathcal{F}(n,\sigma) = \frac{1}{2}\sum_{s=1}^{N_S} \|\sigma u_s u_s^* - H_s\|^2_{L^2(X)} + \beta\mathcal{R}(n,\sigma)$$

  subject to the constraints

$$\begin{aligned} \Delta u_s(\mathbf{x}) + k_s^2(1+n(x))u_s + ik_s\sigma(\mathbf{x})u_s(\mathbf{x}) &= 0, && \text{in } X \\ u_s &= g_s(\mathbf{x}), && \text{on } \partial X \end{aligned}$$

  where N_S is the number of illuminations used, and H_s is the data collected under illumination s.
- The regularization term (with strength β) can vary case by case.

Reconstruction Strategy I: Numerical Optimization

Theorem

Let $z_s = \sigma u_s u_s^* - H_s$ and let w_s be the unique solution of

$$\begin{aligned} \Delta w_s + k_s^2(1+n)w_s + ik_s\sigma w_s &= \sigma z_s u_s^*, \quad \text{in } X \\ w_s &= 0, \qquad\quad \text{on } \partial X \end{aligned}$$

Then the Fréchet derivatives of $\mathcal{F}$ with respective to n and σ are given respectively as

$$\langle \frac{\partial \mathcal{F}}{\partial n}, \hat{n} \rangle = -\sum_{s=1}^{N_S} k_s^2 \langle u_s w_s + u_s^* w_s^*, \hat{n} \rangle$$

$$\langle \frac{\partial \mathcal{F}}{\partial \sigma}, \hat{\sigma} \rangle = \sum_{s=1}^{N_S} \langle z_s u_s u_s^*, \hat{\sigma} \rangle - ik_s \langle u_s w_s - u_s^* w_s^*, \hat{\sigma} \rangle.$$

Reconstruction Strategy I: Numerical Optimization

- Quasi-Newton for Minimization. The optimization problem is solved with a quasi-Newton method using the BFGS updating rule for Hessian, so that no real Hessian need to be calculated.
- PDE Solvers. All elliptic PDEs are solved with a first order finite element method.
- Scaling. Due to the difference between the sensitivity of the solutions to the two parameters, scaling are needed in a few places in the optimization scheme and fixed point iterations
- Multifrequency Data. If $\sigma(\mathbf{x})$ is independent of k in a frequency regime, data with different frequencies can be used in the reconstruction. The reconstruction procedure keeps the same with mixed data.

Reconstruction Strategy II: Fix Point Iteration

Alternatively, we can start with the amplitude-phase formulation

$$\begin{aligned} \Delta|u|_s + k_s^2(1+n)|u|_s &= |u|_s|\nabla\varphi_s|^2, \quad \text{in } X \\ \nabla\cdot|u|_s^2\nabla\varphi_s + k_s H_s &= 0, \quad \text{in } X \\ |u|_s = |g_s| \quad , \quad \varphi_s &= \varphi_{g_s}, \quad \text{on } \partial X \end{aligned}$$

where $1 \le s \le N_S$. We construct the following fixed point iteration:

- Step 0: guess initial value σ_0
- Step j:
  - Solve the elliptic equation for φ_s, $1 \le s \le N_S$

$$\begin{aligned} \nabla\cdot\frac{H_s}{\sigma_j}\nabla\varphi_s + k_s H_s &= 0, \quad \text{in } X \\ \varphi_s &= \varphi_{g_s} \quad \text{on } \partial X \end{aligned}$$

  - Reconstruct $n_j(x) = \frac{1}{N_S}\sum \frac{|u|_s|\nabla\varphi_s|^2 - \Delta|u|_s - k_s^2|u|_s}{k_s^2|u|_s}$.

Reconstruction Strategy II: Fix Point Iteration

- Step $j+1$
  - Solve the nonlinear system for $|u|_s$, $1 \leq s \leq N_S$

$$\begin{aligned} \Delta |u|_s + k_s^2(1+n_j)|u|_s &= |u|_s |\nabla \varphi_s|^2, && \text{in } X \\ \nabla \cdot |u|_s^2 \nabla \varphi_s + k_s H_s &= 0, && \text{in } X \\ |u|_s = |g_s| \quad &, \quad \varphi_s = \varphi_{g_s} && \text{on } \partial X \end{aligned}$$

  - Reconstruct $\sigma_{j+1} = \dfrac{1}{N_S} \sum \dfrac{H_s}{|u|_s^2}$.

Reconstruction Strategy II: Remarks

- Double Iteration. The fixed point iteration is a double iteration since at step $j+1$ we need to solve the nonlinear system by another fixed point iteration as mentioned before.
- Initial Guess. It turns out that the iteration only converges when the intial guess is very close to the true solution.
- Illuminations. In practice, we observe that for a fixed number of illumination, we should choose illuminations that generate very different amplitude and phase functions to produce high quality reconstructions.
- The advantage of this approach is that it is extremely easy to implement. Only a few elliptic solver are needed.

Two Coefficients: Simulation I

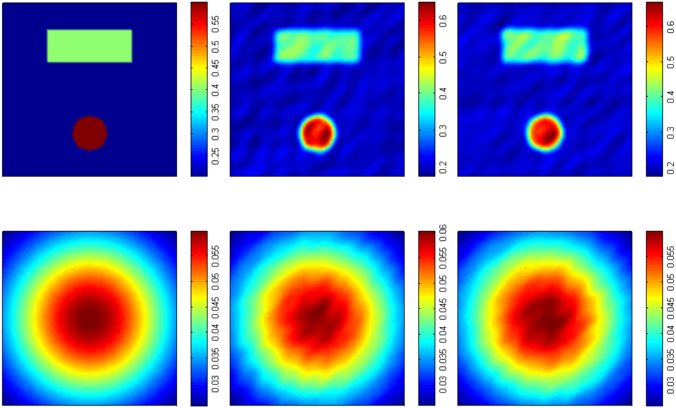

Figure 5: Left to right: real (n, σ), reconstructed (n, σ) with optimization, reconstructed (n, σ) with fix point iteration. Four line sources are used.

Two Coefficients: Simulation II

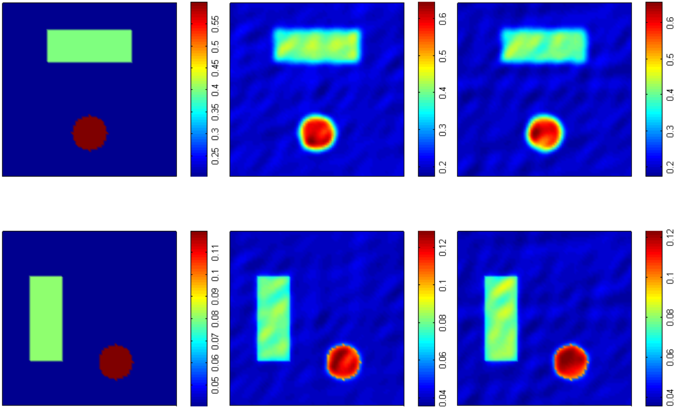

Figure 6: Left to right: real (n, σ), reconstructed (n, σ) with optimization, reconstructed (n, σ) with fix point iteration. Eight sources are used.

Two Coefficients: Simulation III

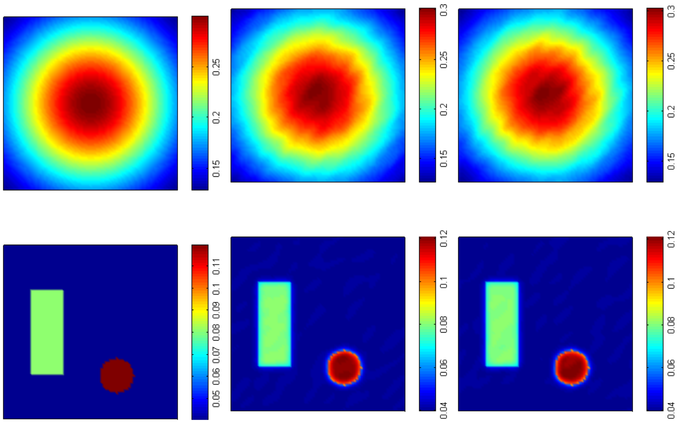

Figure 7: Left to right: real (n, σ), reconstructed (n, σ) with optimization, reconstructed (n, σ) with fix point iteration. Eight line sources (of two different frequencies, assuming $\sigma(\mathbf{x})$ independent of frequency) are used.

Presentation Outline

Concluding Remarks

- We reported some numerical investigation of the inverse Helmholtz problem with interior data aiming at the reconstruction of the conductivity coefficient and refraction index.
- We show that the reconstruction of either coefficient is unique and stable.
- The optimization and fix point algorithm to reconstruct two coefficients converges only when initial guess is very close to true coefficients, usually when

$$\|(n, \sigma)_{guess} - (n, \sigma)_{true}\| / \|(n, \sigma)_{true}\| < 0.1.$$

- Detailed mathematical analysis of the two-coefficient problem (uniqueness and stability) is still missing.