MALTSEV ON TOP

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Abstract. Let $\mathbf{A}$ be an idempotent algebra, $\alpha \in \operatorname{Con} \mathbf{A}$ such that $\mathbf{A}/\alpha$ is Maltsev, and m be a fixed natural number. There is a polynomial time algorithm that can transform any constraint satisfaction problem over $\mathbf{A}$ with relations of arity at most m into an equivalent problem which is m consistent and in which each domain is inside an α block. Consequently if the induced algebras on the blocks of α generate an SD($\wedge$) variety, then CSP($\mathbf{A}$) is tractable.

Let $\mathbf{A}$ and $\mathbf{B}$ be fixed idempotent algebras in the same variety, $\mathbf{B}$ be Maltsev, and m be a fixed and n be an arbitrary natural number.

Lemma 1. *Every subuniverse $\mathbf{S} \leq \mathbf{A}^m \times \mathbf{B}^n$ has a compact representation (polynomially many in n tuples that generate $\mathbf{S}$). There is a polynomial time algorithm that can compute from the compact representations of two subuniverses $\mathbf{S}_1, \mathbf{S}_2$ the compact representation of $\mathbf{S}_1 \cap \mathbf{S}_2$.*

Proof. By an *index* we mean a tuple $(\bar{a}, i, b, c)$ where $\bar{a} \in A^m$, $0 \leq i < n$ and $b, c \in B$. A *witness* for the index $(\bar{a}, i, b, c)$ in a subuniverse $\mathbf{S} \leq \mathbf{A}^m \times \mathbf{B}^n$ is a pair of tuples $(\bar{a}, \bar{b}), (\bar{a}, \bar{c}) \in S$ such that $b_j = c_j$ for all $j < i$ and $b_i = b$ and $c_i = c$. By a *compact representation of* $\mathbf{S}$ we mean a collection of witnesses for all possible indices. We assume that $|A|$, $|B|$ and m are bounded by some fixed constant. Clearly, then every subuniverse $\mathbf{S}$ has a compact representation consisting of polynomially many vectors in n.

We claim that if $\mathbf{S}_1, \mathbf{S}_2 \leq \mathbf{A}^m \times \mathbf{B}^n$ and the compact representations of $\mathbf{S}_1$ and $\mathbf{S}_2$ are known, then the compact representation of $\mathbf{S}_1 \cap \mathbf{S}_2$ can be computed by a polynomial algorithm. To check whether an index $(\bar{a}, i, b, c)$ can be witnessed in $S_1 \cap S_2$ it is enough to check whether (i, b, c) can be witnessed in $P_1 \cap P_2$, as defined by Dalmau, where

$$P_i = \{\, \bar{b} \in B^n : (\bar{a}, \bar{b}) \in S_i \,\}.$$

From his algorithm it is not at all clear that the compact representation of the intersection can be computed at all because in the "next" procedure he has to apply a constraint to a compact representation. There is a trick, however, which can help us here. Suppose that we know the compact representations of P_1 and P_2. Then we can compute the compact representation of $P_1 \times P_2 \leq B^{2n}$, and by applying the equality constraint to the $(1, n+1), \ldots, (n, 2n)$ pairs of indices we get the compact representation of $(P_1 \cap P_2)^2$. This yields the compact representation of $P_1 \cap P_2 \in B^n$ in polynomial time. □

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Lemma 2. *Let $k \leq m$. There is a polynomial time algorithm that can compute from the compact representation of $\mathbf{S} \leq \mathbf{A}^m \times \mathbf{B}^n$ the compact representation of*

$$P = \{ (\bar{a}, \bar{b}) \in A^k \times B^n : \exists \hat{a} \in A^{m-k},\ (\hat{a}, \bar{a}, \bar{b}) \in S \}.$$

Conversely, if the compact representation of $\mathbf{P} \leq \mathbf{A}^k \times \mathbf{B}^n$ is known, then the compact representation of $\mathbf{A}^{m-k} \times \mathbf{P}$ can be computed in polynomial time.

Proof. Take an index $(\bar{a}, i, b, c)$ where $\bar{a} \in A^k$. First we find a tuple $\bar{b} \in B^n$ such that $(\bar{a}, \bar{b}) \in P$ and $b_i = b$. For each choice of $\hat{a} \in A^{m-k}$ we check if S has a witness for $(\hat{a}, \bar{a}, i, b, b)$. If we have no such witness, then $(\bar{a}, i, b, c)$ does not have a witness in P, otherwise we have the tuple $(\hat{a}, \bar{a}, \bar{b}) \in S$ such that $b_i = b$. For each $\tilde{a} \in A$ we can check with the Dalmau algorithm if there exists $\bar{c} \in B^n$ such that $(\tilde{a}, \bar{a}, \bar{c}) \in S$, $c_j = b_j$ for all $j < i$ and $c_i = c$. If we have such, then we are done and $(\bar{a}, \bar{b})$, $(\bar{a}, \bar{c})$ is a witness for $(\bar{a}, i, b, c)$ in P, otherwise there is no wittness in P.

The second claim of the lemma is trivial. □

Definition 3. By a *Maltsev strategy* we mean a collection

$$\mathcal{S} = \{ S_I \leq \mathbf{A}^I \times (\mathbf{A}/\alpha)^n : I \subseteq \{1, \dots, n\},\ |I| \leq m \}$$

of subuniverses such that

(1) $a_i/\alpha = b_i$ for all $(\bar{a}, \bar{b}) \in S_I$ and $i \in I$,
(2) $S_I = \{ (\bar{a}|_I, \bar{b}) : (\bar{a}, \bar{b}) \in S_J \}$ for all $I \subseteq J$.

We say that $\mathcal{S}$ is *nonempty* if $S_\emptyset \neq \emptyset$. A tuple $\bar{u} \in A^n$ is a *solution* of $\mathcal{S}$ if $(\bar{u}|_I, \bar{u}/\alpha) \in S_I$ for all I.

Lemma 4. *Let $\mathcal{P}$ be an instace of* CSP($\mathbf{A}$) *with n variables and constraints of arity less than m. There is a polynomial time algorithm that constructs a Maltsev strategy $\mathcal{S}$ which has a solution if and only if $\mathcal{P}$ does.*

Proof. For all subsets $I \subseteq \{1, \dots, n\}$, $|I| \leq m$, let $P_I \leq A^I$ be the set of partial solutions of $\mathcal{P}$ on the set I of variables. Define

$$S_I = \{ (\bar{a}, \bar{b}) \in P_I \times (\mathbf{A}/\alpha)^n : \forall i \in I,\ a_i/\alpha = b_i \},$$

and put $\mathcal{S} = \{ S_I : I \}$. Clearly, $\mathcal{S}$ has a solution if and only if $\mathcal{P}$ does, however $\mathcal{S}$ is not yet a Maltsev strategy as it might not satisfy condition (2).

Suppose that S_I does not equal $S_J|_I = \{ (\bar{a}|_I, \bar{b}) : (\bar{a}, \bar{b}) \in S_J \}$ for some $I \subset J$. If $S_I \not\subseteq S_J|_I$, then we can replace S_I with $S_I \cap S_J|_I$, which we can calculate by Lemmas 1 and 2. If on the other hand $S_I \not\supseteq S_J|_I$, then we can replace S_J with $S_J \cap (S_I \times A^{J \setminus I})$ where $(S_I \times A^{J \setminus I})$ is the extension of S_I to the domain J. In both cases the number of witnesses decreeses. As therere are only polinomially many witnesses in the whole of $\mathcal{S}$, this consystency step must terminate in polynomially many steps.

Note, that whenever $\mathcal{P}$ has a solution, then the corresponding solution of $\mathcal{S}$ cannot be removed, so $\mathcal{S}$ is nonempty in this case. □

Lemma 5. *If $\mathcal{S}$ is a nonempty Maltsev strategy then for each $(\emptyset, \bar{b}) \in S_\emptyset$ the collection $\mathcal{A} = \{ A_I \leq A^I : I \}$ of the relations*

$$A_I = \{ \bar{a} \in A^I : (\bar{a}, \bar{b}) \in S_I \}$$

is an m-consistent strategy such that $A_{\{i\}} \subseteq b_i \in A/\alpha$ for each $1 \leq i \leq n$.

Proof. Follows directly from the definition. □

Note, that if $\mathbf{A}$ is an idempotent algebra, and $\alpha \in \operatorname{Con} \mathbf{A}$, then the α-blocks are subalgebras of $\mathbf{A}$.

Theorem 6. *Let $\mathbf{A}$ be an idempotent algebra, $\alpha \in \operatorname{Con} \mathbf{A}$ such that $\mathbf{A}/\alpha$ is Maltsev and the α-blocks (which are subalgebras of $\mathbf{A}$) generate* $\mathrm{SD}(\wedge)$ *varieties. If Γ is a finte set of relations over $\mathbf{A}$, that is, $\Gamma \subset \mathsf{SP}(\mathbf{A})$, then* $\mathrm{CSP}(\Gamma)$ *can be solved in polynomial time.*

Proof. Let m be the maximum arity of relations in Γ. By Lemma 4 we can convert any instance $\mathcal{P}$ of $\mathrm{CSP}(\Gamma)$ into an equivalent Maltsev strategy $\mathcal{S}$. If $\mathcal{S}$ is empty, then $\mathcal{S}$ and therefore $\mathcal{P}$ have no solution. On the other hand, if $\mathcal{S}$ is non-empty, then we can take an element $(\emptyset, \bar{b}) \in S_\emptyset$. By Lemma 5 we have a nonempty m-consistent $\mathcal{A} = \{ A_I \leq A^I : I \}$ strategy, such that for each $1 \leq i \leq n$ the domain $A_{\{i\}}$ is a subuniverse of an α-class. We can encode the strategy $\mathcal{A}$ as an instance (or strategy) over the direct product

$$\mathbf{B} = \prod_{\bar{a} \in \alpha} \bar{a}$$

of the α-classes of $\mathbf{A}$. Since the α-classes generate $\mathrm{SD}(\wedge)$ varieties, $\mathbf{B}$ is also generates an $\mathrm{SD}(\wedge)$ variety, therefore by the bounded-width characterization we know that $\mathcal{A}$ has a solution (since it is m-consistent). Therefore $\mathcal{P}$ has a solution as well. □

Corollary 7. *Let $\mathcal{V}$ be a pseudo-variety (containing finite algebras closed under subalgebras, homomorphic images and finite direct products) generated by algebras of bounded width and Maltsev algebras. Then for every algebra $\mathbf{A} \in \mathcal{V}$ and finite set of relations $\Gamma \subset \mathsf{SP}(\mathbf{A})$ the problem* $\mathrm{CSP}(\Gamma)$ *can be solved in polynomial time.*

Proof. We know that $\mathrm{CSP}(\mathbf{A})$ can be solved in polynomial time both for algebras of bounded width and Maltsev algebras. It is true in general that if $\mathrm{CSP}(\mathbf{A})$ can be solved in polynomial time, then the same holds for any subalgebras and homomorphic images of $\mathbf{A}$. So the only problem we face is about finite products.

We know that finite products of bounded width algebras has bounded width as well (from the $\mathrm{SD}(\wedge)$ description). The same tractability result is true for Maltsev algebras, however it is not as straightforward, since the Maltsev operation in these algebras might come from different terms, so the finite direct product might not have a Maltsev term.

One option is to use R. McKenize's observation/result, that the direct product of algebras of few subpowers has few subpowers and reprove all the lemmas in this writeup for few subpowers instead of Maltsev algebras. The other option is to develop a hybrid algorithm for the Maltsev over Maltsev problem, which can be handled somewhat similarly to this writeup.

Finally, we have to show that the direct product of a Maltsev algebra and n algebra of bounded width is stractable. However, in the direct product we do have the α projection congruence which satisfies the conditions of Theorem 6, which finishes the proof. □

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