The model theory of $\mathcal{R}$

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Joint work with I. Farah, I. Goldbring,
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Outline

- How do you know if you are doing model theory?

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- Decidability

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- Ultraproducts!

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- How can you tell if a class might be amenable to a model theoretic study?
- Ultraproducts!
- The key observation is usually identifying a class of structures closed under ultraproducts.

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- The study of II_1 factors via tracial ultraproducts going back to Sakai (and Wright) and McDuff.
- I want to consider II_1 factors as a case study.

Tracial ultraproducts

- Suppose that A is a von Neumann algebra. We say that A is tracial if it has a faithful, normal trace τ i.e. a positive linear functional τ which is both faithful ($\tau(a^*a) = 0$ implies $a = 0$), normal ($\tau(1) = 1$) and satisfies $\tau(xy) = \tau(yx)$. We write $\|a\|_2$ for the usual 2-norm induced by such a trace.

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$$\ell^\infty(\prod_{i\in I} A_i) = \{\bar{a} \in \prod_{i\in I} A_i : \text{for some } M, \|a_i\| \leq M \text{ for all } i \in I\}$$

and

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- The ultraproduct is then $\prod_{i \in I} A_i / U := \ell^\infty(\prod_{i \in I} A_i)/c_U$.

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- The functions we highlight (those “in the language") are all *-polynomials which map the operator norm unit ball of *any* tracial von Neumann algebra back into itself. For instance, $\frac{x+y}{2}, xy, x^*, \lambda x$ for $|\lambda| \leq 1$ etc.

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- We also highlight the real and imaginary parts of the fixed trace.
- One key element of the general theory of metric structures is that all of these functions and relations are uniformly continuous with respect to the 2-norm.

The logic of metric structures

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- Arbitrary formulas are obtained by “quantifying" over the variables using either sup or inf over the operator norm unit ball.
- So an arbitrary formula has the form:

$$Q^1_{x_1 \in B_1} Q^2_{x_2 \in B_1} \ldots Q^k_{x_k \in B_1} \varphi(x_1, \ldots, x_n)$$

  where each Q^i is either sup or inf and φ is a quantifier-free formula.

Interpreting formulas

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- The theory of an algebra A in continuous logic is the function from sentences φ to numbers φ^A which assigns their value in A; we write $Th(A)$ for this function.
- It is equivalent to determine the set of sentences in a given algebra which evaluate to 0. In fact, we can determine $Th(A)$ from knowing the zero set on positive sentences.

Elementary classes

We say that a class of structures K is elementary if there is a set of sentences T such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

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1. *The class of all tracial von Neumann algebras is elementary.*
2. *The class of all II_1 factors is elementary.*

Elementary maps

If $A \subseteq B$ are two algebras then we say this embedding is elementary if for all formulas $\varphi(\bar{x})$ and $\bar{a} \in A$, $\varphi^A(\bar{a}) = \varphi^B(\bar{a})$.

Theorem (Łoś Theorem)

Suppose A_i are tracial von Neumann algebras for all $i \in I$, U is an ultrafilter on I, $\varphi(\bar{x})$ is a formula and $\bar{a} \in \prod_{i \in I} A_i/U$ then

$$\varphi(\bar{a}) = \lim_{i \to U} \varphi^{A_i}(\bar{a}_i)$$

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It follows that the diagonal embedding of A into A^U is always elementary; in particular, $Th(A) = Th(A^U)$.

Property Γ

- If A is a II_1 factor and U is a non-principal ultrafilter on $\mathbb{N}$, a relative commutant of A in A^U, written $A' \cap A^U$ is

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- Having property Γ is independent of the choice of ultrafilter. In fact, property Γ is an elementary property.
- Indeed the sentences φ_n, for all $n \in N$, express property Γ where φ_n is

$$\sup_{x_1,\ldots,x_n \in B_1} \inf_{y \in B_1} \left(\sum_{i=1}^{n} \|[x_i, y]\|_2 + |\tau(y)| + \|1 - y^* y\|_2\right)$$

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Corollary ($\neg$ CH)

If A is McDuff then it has non-isomorphic relative commutants.

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- One common reason model theoretically for this behaviour is that the given theory has quantifier elimination i.e. for any formula $\varphi(\bar{x})$ and $\epsilon > 0$ there is a quantifier-free formula $\psi(\bar{x})$ such that

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- So, does $Th(\mathcal{R})$ have quantifier elimination?

Quantifier complexity, cont'd

- No! A paper of Nate Brown's contains the following calculation: If $\Gamma = SL_3(Z) * Z$ then $L(\Gamma)$ is $\mathcal{R}^\omega$-embeddable; in fact it has an automorphism α and embedding $\pi : L(\Gamma) \to \mathcal{R}^\omega$ for which α is not implemented by a unitary.

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Theorem (Goldbring, H., Sinclair)

If $Th(\mathcal{R})$ is model complete then CEP fails!

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- General fact: If $A \subseteq B$ then $Th_\forall(B) \subseteq Th_\forall(A)$.
- $\mathcal{R} \hookrightarrow A$ for any II_1 factor so $Th_\forall(A) \subseteq Th_\forall(\mathcal{R})$.
- $\mathcal{R} \prec \mathcal{R}^\omega$ so $Th_\forall(\mathcal{R}) = Th_\forall(\mathcal{R}^\omega)$. It follows then that CEP holds iff $Th_\forall(A) = Th_\forall(\mathcal{R})$ for all II_1 factors A.

Microstate conjecture

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- It is immediate that CEP holds iff the microstate conjecture is true i.e. For any II_1 factor A, $\epsilon > 0$, *-polynomials $p_1(\bar{x}), \ldots, p_n(\bar{x})$ and $\bar{a} \in A$ there is $\bar{b} \in \mathcal{R}$ (alternatively, there is N and $\bar{b} \in M_N$) such that for all $i = 1, \ldots, n$,

$$|tr(p_i(\bar{a})) - tr(p_i(\bar{b}))| \leq \epsilon$$

Even without CEP

- $Th_\forall(\mathcal{R})$ is maximal among universal theories of II_1 factors; it follows by Łoś' theorem that there is a minimal universal theory i.e. there is a separable II_1 factor $\mathcal{S}$ such that for all II_1 factors A, $Th_\forall(\mathcal{S}) \subseteq Th_\forall(A)$.

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- Again, it is immediate that for any separable II_1 factor A, $A \hookrightarrow \mathcal{S}^\omega$ (a poor man's resolution to CEP).
- Note: $Th_\forall(\mathcal{S}) = Th_\forall(\mathcal{R})$ iff CEP holds.

Even without CEP

- $Th_\forall(\mathcal{R})$ is maximal among universal theories of II_1 factors; it follows by Łoś' theorem that there is a minimal universal theory i.e. there is a separable II_1 factor $\mathcal{S}$ such that for all II_1 factors A, $Th_\forall(\mathcal{S}) \subseteq Th_\forall(A)$.
- Again, it is immediate that for any separable II_1 factor A, $A \hookrightarrow \mathcal{S}^\omega$ (a poor man's resolution to CEP).
- Note: $Th_\forall(\mathcal{S}) = Th_\forall(\mathcal{R})$ iff CEP holds.
- Good question: what could $\mathcal{S}$ look like?

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- If CEP holds the same would be true if universal was replaced by existential in the previous statement and with some additional work, one can even get an algorithm that approximates values for $\exists\forall$-sentences.
- CEP is equivalent to the decidability of the universal theories of all type II_1 tracial von Neumann algebras.