Okubo algebras in characteristic 3 and valuations

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Introduction

1978: Okubo defines Okubo algebras (pseudo-octonion algebra) over $\mathbb{C}$.
1990: Classification of Okubo algebras in characteristic different from 3 by Elduque and Myung.
1996: Conceptual definition of pseudo-octonion algebra by Elduque and Pérez.
1997: Classification of Okubo algebras in characteristic 3 by Elduque.

Elduque describes Okubo algebras in characteristic 3 as a "limit" of Okubo algebras in other characteristics.

Goal: To confirm this behavior by the use of valuations.

Outline

1. Preliminaries
   - Okubo algebras
   - Valuations on division algebras
2. Residue of Okubo algebras

Okubo algebras

Definition

An Okubo algebra A over a field F is a (non unital, non associative) algebra over F such that $A \otimes_F F_{\mathrm{alg}}$ is the pseudo-octonion algebra over F_{alg} (i.e. the split octonion algebra with a twisted product).

Okubo algebras are *symmetric composition algebras*, i.e. algebras with a (unique) non-degenerate quadratic form n, called the *norm*, such that $x(yx) = (xy)x = n(x)y$ for all x, y.

Classification over F with $\mathrm{char}(F) \neq 3$ and $F \ni \omega$

Theorem (Elduque and Myung)

Assume that $\mathrm{char}(F) \neq 3$ *and* F *contains a primitive cube root of unity* ω*. Okubo algebras over* F *are the* $(A^0, *)$ *where*

- A *is a degree 3 central simple algebra over* F
- $A^0 = \{x \in A \mid \mathrm{Trd}(x) = 0\}$, Trd *being the reduced trace of* A
- $x * y = \frac{1-\omega}{3}(xy - \omega^2 yx) - \frac{1}{3}\mathrm{Trd}(xy) \cdot 1$ *(Okubo product)*

Classification over F with $\mathrm{char}(F) \neq 3$ and $F \not\ni \omega$

Theorem (Elduque and Myung)

Assume that $\mathrm{char}(F) \neq 3$ and F does not contain a primitive cube root of unity $\omega \in F_{\mathrm{alg}}$.
Okubo algebras over F are the $(\mathrm{Sym}(A)^0, *)$ where

- *A is a degree 3 central simple algebra over $F(\omega)$*
- *τ is an $F(\omega)/F$-involution of the second kind on A*
- $\mathrm{Sym}(A)^0 = \{x \in A \mid \tau(x) = x$ *and* $\mathrm{Trd}(x) = 0\}$
- *$*$ is the Okubo product*

Classification over F with $\mathrm{char}(F) = 3$

For $\alpha, \beta \in F^\times$, consider $F[a,b] = F[X,Y]/(X^3 - \alpha, Y^3 - \beta)$, where a and b are the images of X and Y, and define $C_{\alpha,\beta}$ as the subspace

$$C_{\alpha,\beta} = \mathrm{span}\langle a^i b^j \mid 0 \le i,j \le 2, (i,j) \neq (0,0)\rangle \subset F[a,b].$$

Theorem (Elduque)

Assume that $\mathrm{char}(F) = 3$. *Okubo algebras over* F *are the* $(C_{\alpha,\beta}, *)$ *where*

- $\alpha, \beta \in F^\times$
- $\diamond$ *on* $F[a,b]$ *is defined by* $a^i b^j \diamond a^{i'} b^{j'} = \left(1 - \begin{vmatrix} i & j \\ i' & j' \end{vmatrix}\right) a^{i+i'} b^{j+j'}$
- $x * y \in C_{\alpha,\beta}$ *is the unique element such that* $x \diamond y = b(x,y) + x * y$ *with* $b(x,y) \in F$

Valuations on division algebras

Let D be a division ring, Γ a totally ordered abelian group and ∞ a symbol.

Definition

A map $v\colon D \to \Gamma \cup \{\infty\}$ is a *valuation* on D if

(i) $v(x) = \infty$ if and only if $x = 0$

(ii) $v(xy) = v(x) + v(y)$

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$

$\Gamma_D := v(D^\times)$ *value group*
$\mathscr{O}_D := \{x \in D \mid v(x) \geq 0\}$ *valuation ring*
$\mathfrak{m}_D := \{x \in D \mid v(x) > 0\}$ maximal ideal
$\overline{D} := \mathscr{O}_D/\mathfrak{m}_D$ *residue division ring*, $\mathscr{O}_D \to \overline{D}\colon x \mapsto \overline{x}$

Height of a division algebra

Assume that D is a degree p central division algebra over a field F such that $\mathrm{char}(\overline{F}) = p$ and $[\overline{D}\colon \overline{F}] \cdot (\Gamma_D\colon \Gamma_F) = p^2$ (D is defectless).

Definition (Tignol)

The *height* of D is $h(D) := \min\{v(xy - yx) - v(xy) | x, y \in D\}$.

One has $0 \leq h(D) \leq v(p)/(p-1)$.

Residue of Okubo algebras

Let F be a field, $\mathrm{char}(F) = 0$, and let S be an Okubo algebra over F. So S is either $(D^0, *)$ or $(\mathrm{Sym}(D)^0, *)$ for some degree 3 central simple algebra D over $F(\omega)$.

Assume furthermore that D is a division algebra (i.e. S does not contain nonzero idempotents) and that D is endowed with a valuation v such that $\mathrm{char}(\overline{F}) = 3$.

Goal: To give a criterion for $(\overline{S}, \overline{*})$ to be an Okubo algebra over $\overline{F}$ where

$$\overline{S} := \{\overline{x} \in \overline{D} \mid x \in \mathscr{O}_D \cap S\} \text{ and } \overline{x} \,\overline{*}\, \overline{y} = \overline{x * y}.$$

First case: $\omega \in F$, $S = (D^0, *)$

- $[\overline{D}\colon \overline{F}] \cdot (\Gamma_D\colon \Gamma_F)$ divides $[D\colon F]$ (Morandi),
  so if $(\overline{D^0}, \overline{*})$ is an Okubo algebra over $\overline{F}$ then $[\overline{D}\colon \overline{F}] = 9$.
- If $[\overline{D}\colon \overline{F}] = 9$ and $h(D) < v(3)/2$, then $*$ does not restrict to $\mathscr{O}_D \cap D^0$.

Theorem (R.)

Let F be a field, $\mathrm{char}(F) = 0$, $\omega \in F$. Assume that D is a degree 3 central division algebra over F with valuation v such that $\mathrm{char}(\overline{F}) = 3$. Then $(\overline{D^0}, \overline{*})$ is an Okubo algebra over $\overline{F}$ if and only if $[\overline{D}\colon \overline{F}] = 9$ and $h(D) = v(3)/2$.

Sketch of proof

The conditions $[\overline{D}\colon \overline{F}] = 9$ and $h(D) = v(3)/2$ imply that:

- $\overline{*}$ is well-defined;
- the dimension of $\overline{D^0}$ over $\overline{F}$ is equal to 8;
- $(\overline{D^0}, \overline{*})$ is a symmetric composition algebra over $\overline{F}$ (with norm $\overline{n}$), so $\overline{D^0}$ is either an Okubo algebra or a para-octonion algebra;
- $g(\overline{x}) = b_{\overline{n}}(\overline{x}, \overline{x} \mathbin{\overline{*}} \overline{x}) = \overline{x}^3$ for all $\overline{x} \in \overline{D^0}$.

Facts:

- A para-octonion algebra always contains a nonzero idempotent;
- a symmetric composition algebra contains a nonzero idempotent if and only if $g\colon x \mapsto b_n(x, x * x)$ is isotropic.

Second case: $\omega \notin F$, $S = (\mathrm{Sym}(D)^0, *)$

Theorem (R.)

Let F be a field, $\mathrm{char}(F) = 0$, $\omega \notin F$.
Assume that D is a degree 3 central division algebra over $F(\omega)$, τ an $F(\omega)/F$-involution on D, and D is endowed with a valuation v such that $\mathrm{char}(\overline{F}) = 3$.
Then $(\overline{\mathrm{Sym}(D)^0}, \overline{*})$ is an Okubo algebra over $\overline{F}$ if and only if $[\overline{D} \colon \overline{F(\omega)}] = 9$ and $h(D) = v(3)/2$.

Sketch of proof: One can show that $\overline{\mathrm{Sym}(D)^0} \otimes_{\overline{F}} \overline{F(\omega)}$ embeds in $\overline{D^0}$. The conditions $[\overline{D} \colon \overline{F(\omega)}] = 9$ and $h(D) = v(3)/2$ imply that $\overline{D^0}$ is an Okubo algebra over $\overline{F(\omega)}$ and the dimension of $\overline{\mathrm{Sym}(D)^0}$ over $\overline{F}$ is equal to 8.

Okubo algebras in characteristic 3 are always a residue

Theorem (R.)

Let k be a field, $\mathrm{char}(k) = 3$, and let S_0 be an Okubo k-algebra without nonzero idempotents.

1. *There exist a field F, $\mathrm{char}(F) = 0$, $\omega \in F$, and a degree 3 central division algebra D over F with valuation v such that $\overline{F} = k$ and $(\overline{D^0}, \overline{*}) \cong S_0$.*
2. *There exist a field F, $\mathrm{char}(F) = 0$, $\omega \notin F$, a degree 3 central division algebra D over $F(\omega)$ with valuation v and an $F(\omega)/F$-involution τ on D such that $\overline{F} = k$ and $(\overline{\mathrm{Sym}(D)^0}, \overline{*}) \cong S_0$.*

Sketch of proof: There exists a Henselian valued field F such that $\overline{F} = k$. Let $\lambda, \mu \in k^\times$ be such that $S_0 = C_{\lambda,\mu}$. Take $\alpha, \beta \in F$ such that $\overline{\alpha} = \lambda$, $\overline{\beta} = \mu$, then one can choose $D := (\alpha, \beta)_{F(\omega),\omega}$.

THE END