Massive algebras

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(all uncredited results are due to some subset of {I.F., B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Toms, W. Winter}.)

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Notation

A: a separable C*-algebra or (in most of the results) a II_1 factor with a separable predual.

$\mathcal{U}$: a nonprincipal ultrafilter on $\mathbb{N}$.

Massive algebras

$A^{\mathcal{U}}$ is the ultrapower of A,

$$\ell_\infty(A)/c_{\mathcal{U}}(A)$$

where

$$c_{\mathcal{U}}(A) = \{a \in \ell_\infty(A) : \lim_{n\to\mathcal{U}} \|a_n\| = 0\}.$$

$$\bigoplus_{\mathbb{N}}(A) = \{a \in \ell_\infty(A) : \lim_{n\to\infty} \|a_n\| = 0\}.$$

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Via the diagonal embedding, we identify A with a subalgebra of $A^{\mathcal{U}}$ or a subalgebra of $\ell_\infty(A)/\bigoplus_{\mathbb{N}}(A)$.
Ultrapowers are well-studied in logic and all of their important properties follow from two basic principles. Only one of them (countable saturation) is shared by $\ell_\infty(A)/\bigoplus_{\mathbb{N}}(A)$.

The *relative commutant* is

$$A' \cap A^{\mathcal{U}} = \{b : ab = ba \text{ for all } a \in A\}.$$

This is isomorphic to

$$F(A) = A' \cap A^{\mathcal{U}} / Ann(A, A^{\mathcal{U}})$$

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There is no known abstract analogue of relative commutant in model theory in general.

Massive algebras

An algebra C is *countably quantifier-free saturated* if for every sequence of *-polynomials $p_n(x_1, \ldots, x_n)$ with coefficients in C and $r_n \in [0, 1]$ the system

$$\|p_n(a_1, \ldots, a_n)\| = r_n$$

has a solution in C whenever every finite subset has an approximate solution in C.

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Applications of saturation

Proposition (Choi–F.–Ozawa, 2013)

Assume A is countably degree-1 saturated and Γ is a countable amenable group. Then every uniformly bounded representation $\Phi\colon \Gamma \to GL(A)$ is unitarizable.

Discontinuous functional calculus

Proposition

Assume C is countably degree-1 saturated,

1. *$a \in C$ is normal,*
2. *$B \subseteq \{a\}' \cap C$ is separable,*
3. *$U \subseteq \mathrm{sp}(a)$ is open, and*
4. *$g\colon U \to \mathbb{C}$ is bounded and continuous.*

Then there exists $c \in C^*(B, a)' \cap C$ such that for every $f \in C_0(U \cap \mathrm{sp}(a))$ one has

$$cf(a) = (gf)(a).$$

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Brown–Douglas–Fillmore' Second Splitting Lemma

is the special case when $C = B(H)/K(H)$, $\mathrm{sp}(a) = [0, 1]$, and $g(x) = 0$ if $x < 1/2$ and $g(x) = 1$ if $x > 1/2$.

Strongly self-absorbing (s.s.a.) C*-algebras

Definition (Toms–Winter)

A separable algebra A is *s.s.a.* if

1. $A \cong A \otimes A$,
2. The isomorphism between A and $A \otimes A$ is approximately unitarily equivalent with the map $a \mapsto a \otimes 1_A$.

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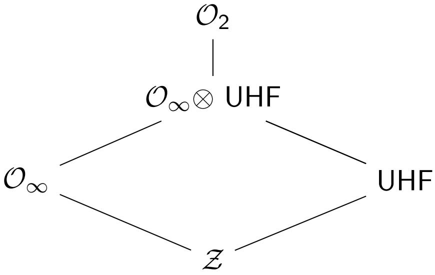
Lemma

Assume A is s.s.a.

1. *(Connes) If A is a II_1 factor, then $A \cong R$.*
2. $A \cong \bigotimes_{\aleph_0} A$.
3. *(Effros–Rosenberg, 1978) If A is a C*-algebra, then A is simple and nuclear.*

All known s.s.a. C*-algebras



Proposition (McDuff, Toms–Winter)

Assume D is s.s.a.. Then for a separable A the following are equivalent.

(i) $A \otimes D \cong A$.

(ii) *There is a unital *-homomorphism from D into $A' \cap A^{\mathcal{U}}$.*

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Theorem (Ghasemi, 2013)

Every countably degree-1 saturated algebra is tensorially prime. In particular, Calkin algebra is tensorially prime and $A^{\mathcal{U}} \otimes D \not\cong A^{\mathcal{U}}$ for any infinite-dimensional A and U.

All ultrafilters are nonprincipal ultrafilters on $\mathbb{N}$

Question (McDuff 1970, Kirchberg, 2004)

Assume A is separable. Does $A' \cap A^{\mathcal{U}}$ depend on $\mathcal{U}$?

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Proposition

If A is a commutative tracial von Neumann algebra, then $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ for all nonprincipal ultrafilters $\mathcal{U}$, $\mathcal{V}$ on $\mathbb{N}$.

Proof.

By Maharam's theorem, $A^{\mathcal{U}} \cong L_\infty(2^{2^{\aleph_0}}, \text{Haar measure})$. □

Theorem (Ge–Hadwin, F., F.–Hart–Sherman, F.–Shelah)

Assume A is a separable C-algebra or a II_1-factor with a separable predual.*
If Continuum Hypothesis (CH) holds then $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ and $A' \cap A^{\mathcal{U}} \cong A' \cap A^{\mathcal{V}}$ for all nonprincipal ultrafilters $\mathcal{U}$, $\mathcal{V}$ on $\mathbb{N}$.

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If CH fails and A is infinite-dimensional, then

1. *there are $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers of A and*
2. *there are $2^{2^{\aleph_0}}$ nonisomorphic relative commutants of A.*

CH is a red herring

Two C*-algebras C_1 and C_2 have the *countable back-and-forth property* if there exists a family $\mathcal{F}$ with the following properties.

1. Each $f \in \mathcal{F}$ is a *-isomorphism from a separable subalgebra of C_1 into C_2.
2. If $\{f_n : n \in \mathbb{N}\}$ is a $\subseteq$-increasing chain in $\mathcal{F}$ then $\bigcup_n f_n \in \mathcal{F}$.
3. If $f \in \mathcal{F}$, $a \in C_1$ and $b \in C_2$ then there is $g \in \mathcal{F}$ such that $g \supseteq f$, $a \in dom(g)$ and $b \in range(g)$.

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Lemma

Assume C_1 and C_2 have the countable back-and-forth property and each one has a dense subset of cardinality $\aleph_1$.
Then they are isomorphic.

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CH $\Leftrightarrow$ $A^{\mathcal{U}}$, $A' \cap A^{\mathcal{U}}$ has a dense subset of cardinality $\aleph_1$ for all separable A.

One of my favourite open problems

Let s denote the image of the unilateral shift in the Calkin algebra $B(H)/K(H)$.

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Question (Brown–Douglas–Fillmore)

Is there an automorphism of $B(H)/K(H)$ *that sends* s *to* s^**?*

Theorem (F., 2007)

There is a model of ZFC in which all automorphisms of $B(H)/K(H)$ are inner, in particular no automorphism sends s to s^*.

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Is there a countable back-and-forth property $\mathcal{F}$ for $B(H)/K(H)$, $B(H)/K(H)$ such that $f(s) = s^*$ for all $f \in \mathcal{F}$?

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The answer to this question is unlikely to be independent from ZFC.
Under CH, a positive answer is equivalent to the positive answer to the BDF question.

Theorem

Assume Continuum Hypothesis. Let D be s.s.a.. Then

$$D' \cap D^{\mathcal{U}} \cong D^{\mathcal{U}}$$

and

$$D' \cap \ell_\infty(D)/\bigoplus_{\mathbb{N}}(D) \cong \ell_\infty(D)/\bigoplus_{\mathbb{N}}(D).$$

Theorem

*Assume C is countably saturated, D is s.s.a., and that there is a unital *-homomorphism from D into $X' \cap C$ for every separable X. Then*

1. *Any two unital *-homomorphisms of D into C are unitarily conjugate.*
2. *Algebras C and $D' \cap C$ have the countable back-and-forth property.*

Proposition

*Assume D is $\mathcal{O}_2$ or UHF and that CH holds. Then there is a unital *-homomorphism*

$$\Phi\colon \bigotimes_{\aleph_1} D \to D^{\mathcal{U}}$$

such that the relative commutant of its range is trivial.

Concluding remarks

Theorem (F.–Shelah, 2014)

The corona of $C([0,1))$ *is countably saturated, but the corona of* $C(Y)$ *for some one-dimensional, locally compact subset of* $\mathbb{R}^2$ *is not.*

Concluding remarks

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Question

Is the corona of $C(\mathbb{R}^n)$ *countably saturated for* $n \geq 2$*?*

For more information see CJ Eagle, A Vignati, arXiv:1406.4875, 2014.