Experimental measurement of a point in phase-space:

Observing Dirac's classical analog to the quantum state

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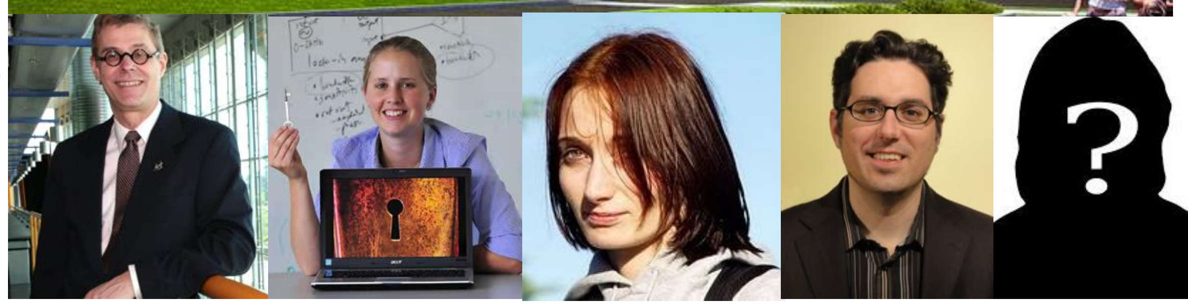


Bob Boyd | Anne Broadbent | Ksenia Dolgaleva | Jeff Lundeen | At least one more

Recruiting undergrads, graduate students, and post-docs

www.photonicquantum.info for more information

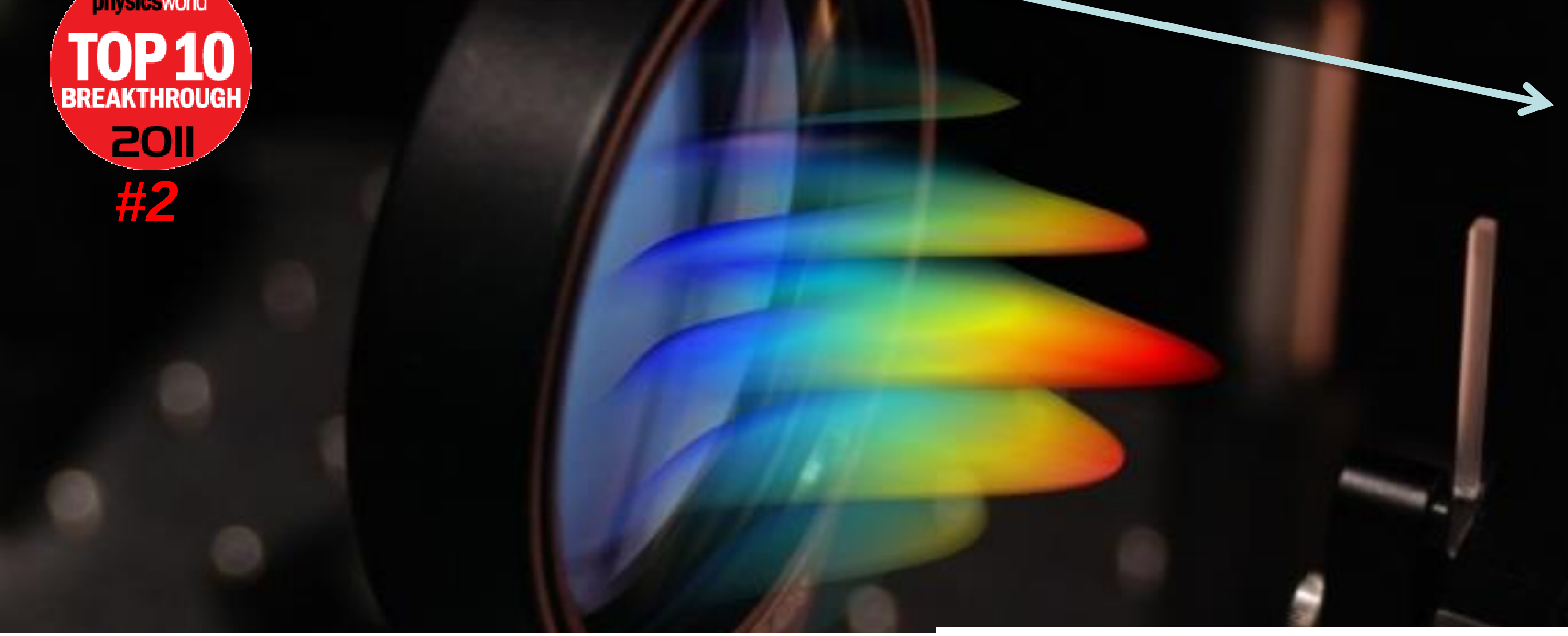
Direct measurement of the wavefunction

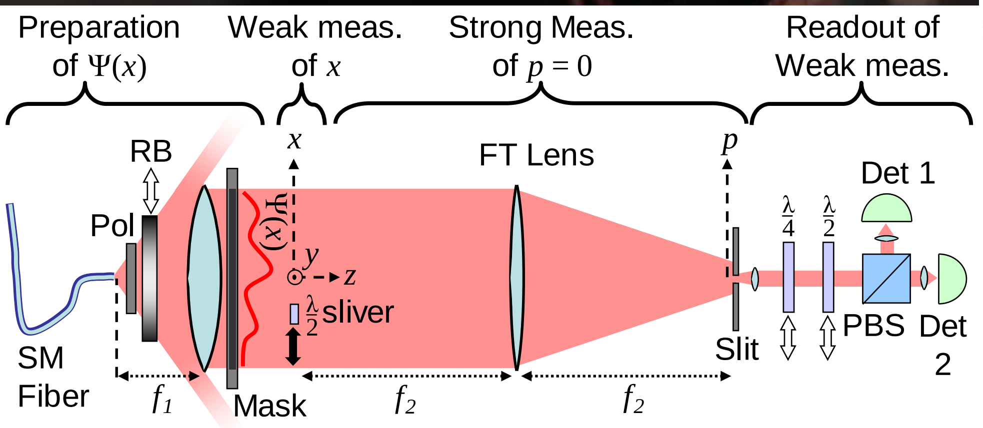
Jeff Lundeen, B. Sutherland, C. Stewart, A. Patel, C. Bamber

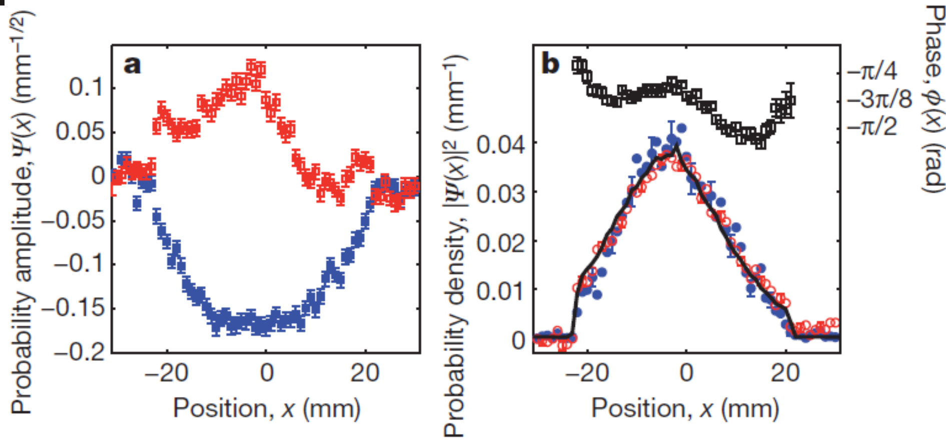
Nature, 474, 188 (2011).

physicsworld TOP 10 BREAKTHROUGH 2011

#2



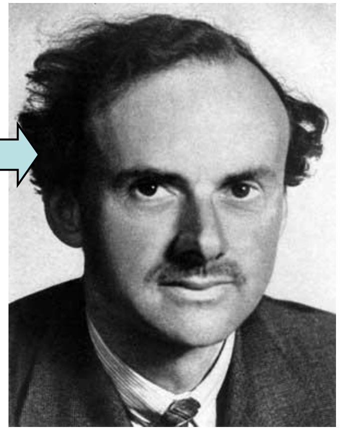




Can this direct procedure be generalized to mixed quantum states?

Dirac's Distribution

José Moyal re-invented the Wigner function



Paul Dirac thought it was a poor idea.

REVIEWS OF MODERN PHYSICS VOLUME 17, NUMBERS 2 AND 3 APRIL–JULY, 1945

On the Analogy Between Classical and Quantum Mechanics

P. A. M. Dirac
St. John's College, Cambridge, England

$$D_\rho(x,p) = \langle p||x\rangle\langle x|\, \boldsymbol{\rho}\, |p\rangle$$

(But first discussed by McCoy in 1932)

- In physics, the Dirac Distribution was forgotten as a theoretical novelty (There was no way to measure it!)

The distribution is complex!

- The Solution is *Weak Measurement*:
- We call the average result of a joint weak-strong A-B measurement the ***weak average*** $= \langle BA \rangle$

$$= \mathrm{Tr}[|p\rangle\langle p||x\rangle\langle x|\, \boldsymbol{\rho}] = D_\rho(x,p)$$

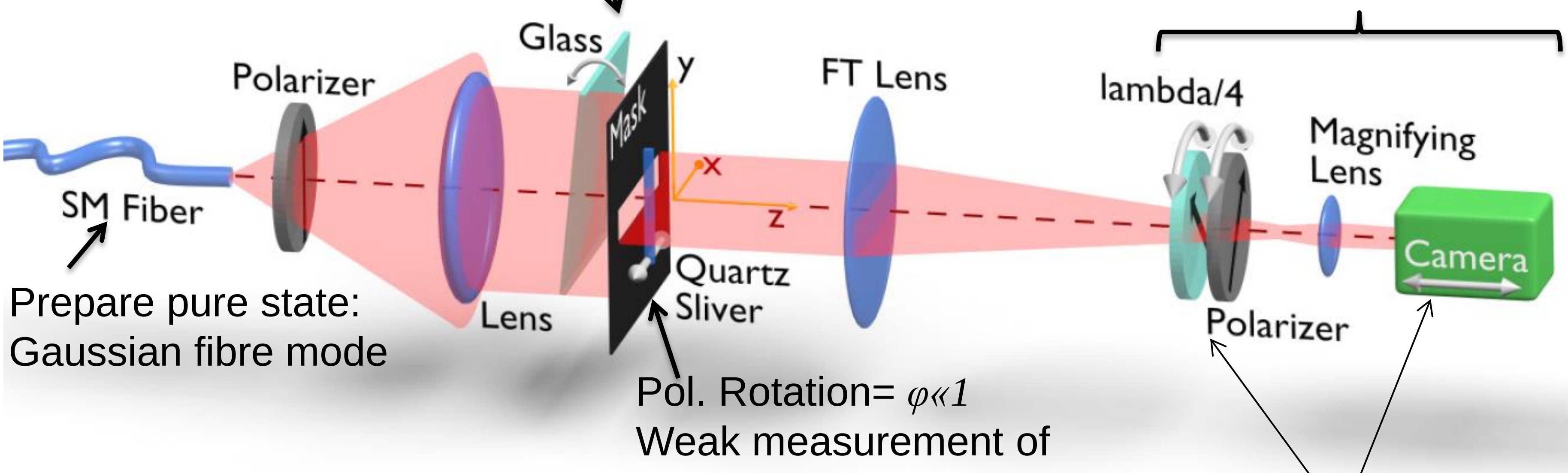
Lundeen, Bamber, PRL 108, 070402 (2012)

Measurement of the Dirac Distribution

- We measured the transverse state of a photon
- Make a weak-strong joint measurement of X and P
- For each x measure all p with an array.

Transform to a mixed state:
Vibrating Glass Plate

Joint readout of x and p
measurements



Prepare pure state:
Gaussian fibre mode

Pol. Rotation= *φ«1*
Weak measurement of
transverse position |x⟩⟨x|

$$D_\rho(x,p)\sin\varphi = \frac{N_{p,45^\circ} - N_{p,-45^\circ}}{N_T} - i\,\frac{N_{p,\circlearrowright} - N_{p,\circlearrowleft}}{N_T}$$

- Not a weak value (not post-selected) but still complex

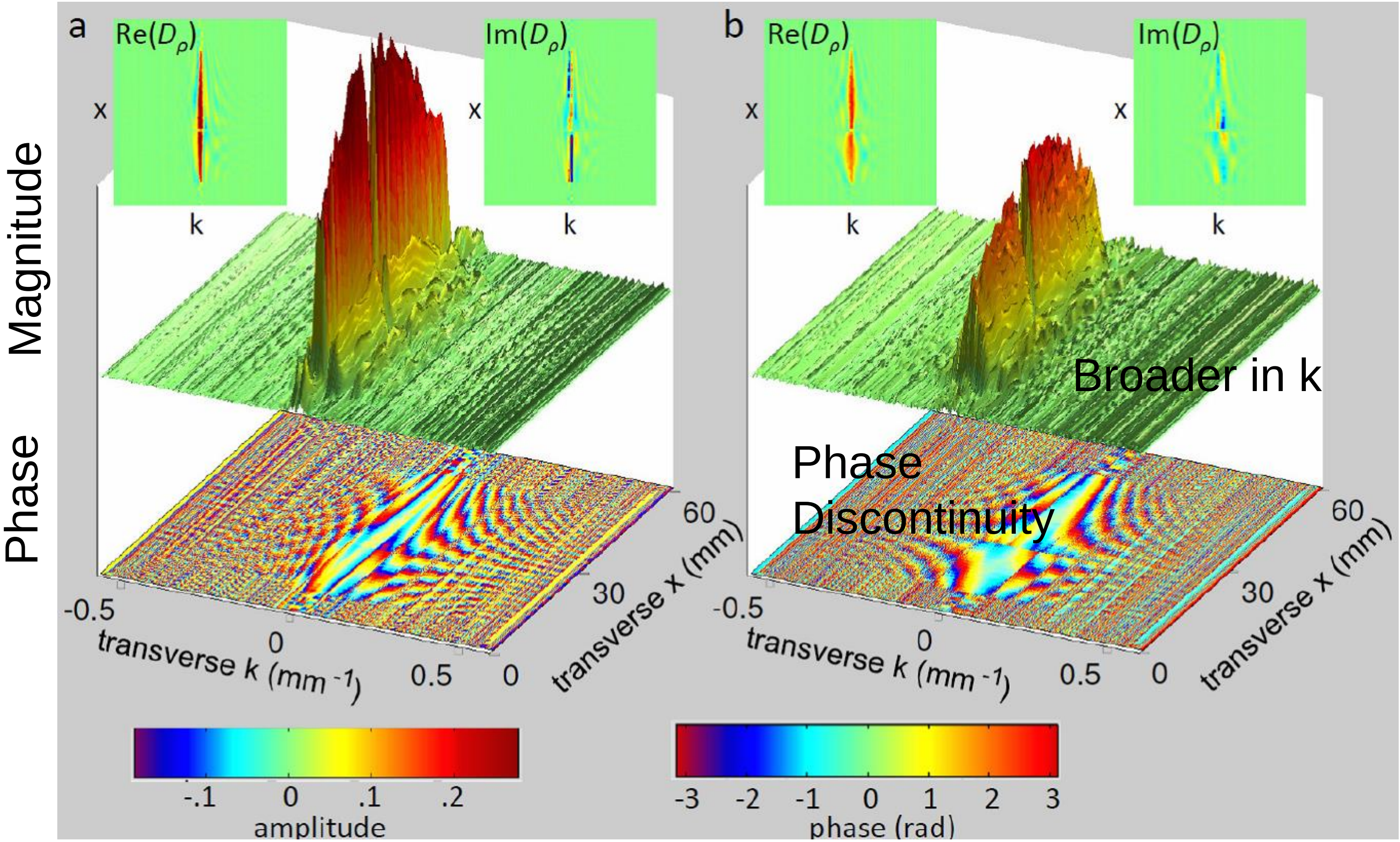
Experimental Dirac Distributions, D_ρ

Pure State

$$D_\rho = \Psi(x)\Phi^*(p)\exp(ipx/\hbar)$$

Mixed State

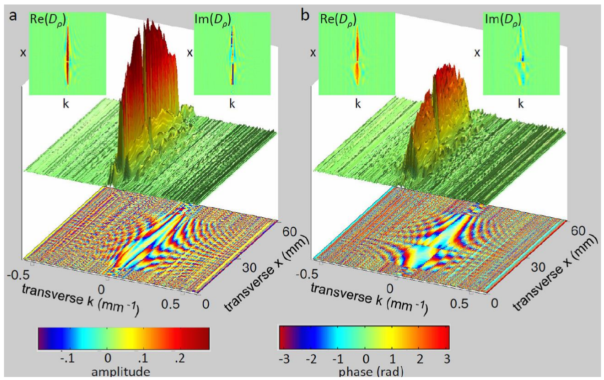
$$D_\rho = [\sum \Psi_j(x)\Phi^*_j(p)]\cdot\exp(ipx/\hbar)$$



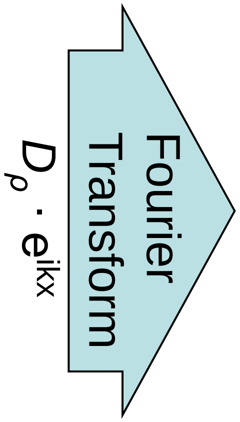
- The Dirac distribution can represent both pure and mixed states

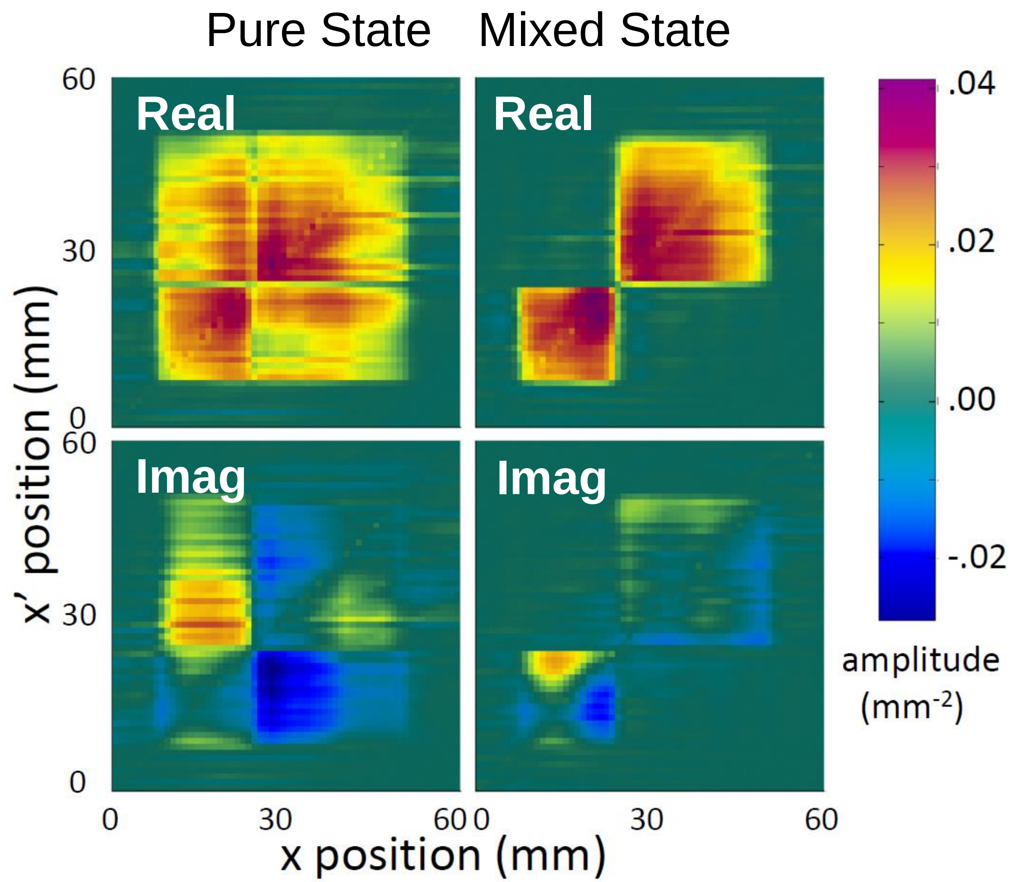
Relationship to the Density Matrix

Measured Dirac Distributions



Pure State Mixed State





- The density matrices are approx. Hermitian (not guaranteed)
- The off-diagonals between glass and no glass are zero
    - The state exhibits no coherence between the two regions

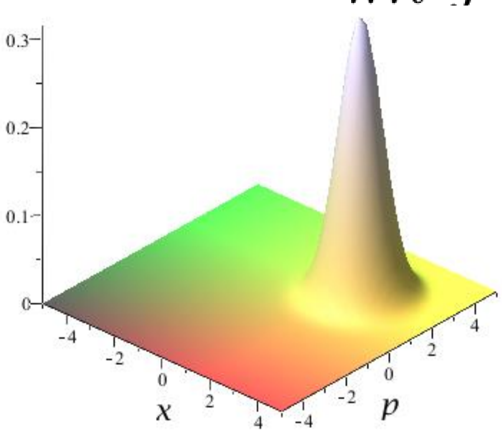
Quasi-Probability Distributions

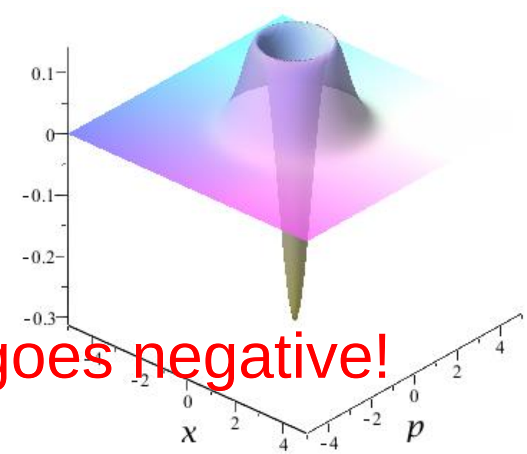
- In classical physics we have the *Liouville* Distribution, Prob(x,p), a phase space (i.e. position-momentum) distribution for an ensemble of particles.
- Any quantum analog will not satisfy some of the standard laws of probability (e.g. Prob>0)

  → ***Quasi-Probability*** Distribution

- 1932, Eugene Wigner: **Wigner Function**

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \langle x+y|\hat{\rho}|x-y\rangle e^{-2ipy/\hbar}\, dy$$



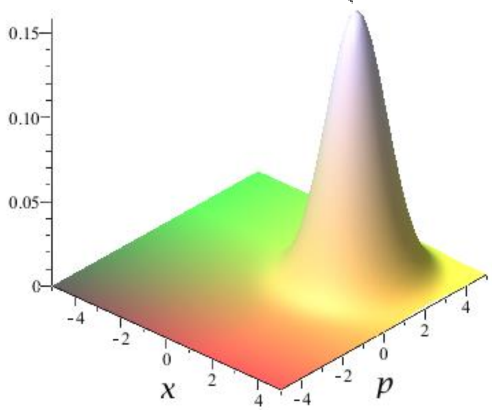


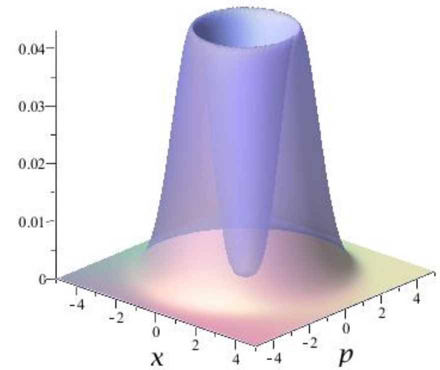
It goes negative!

Other Quasi-Probability Distributions

- 1940, Kodi Husimi: **Q function**

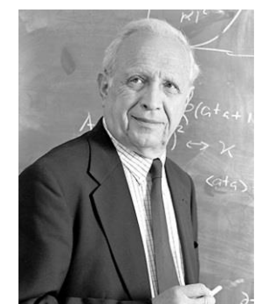
$$Q(\alpha = x + ip) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$$



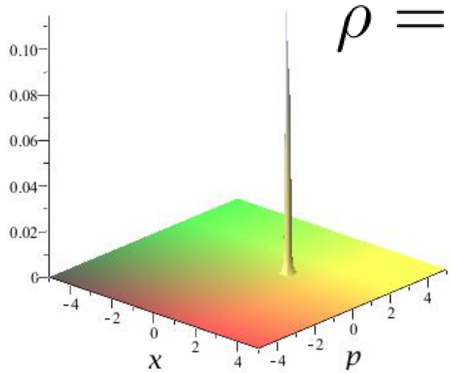


Marginals are not correct, e.g. $\int Q(x,p)\mathrm{d}p \neq \mathrm{Prob}(x)$

- 1963: R. Glauber, G. Sudarshan: **P function**



$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha,$$



$P(x,p)$ is highly singular for most non-classical states

An issue of how to quantize phase-space

- The Q-function, Wigner function, and P-function reflect different operator orderings

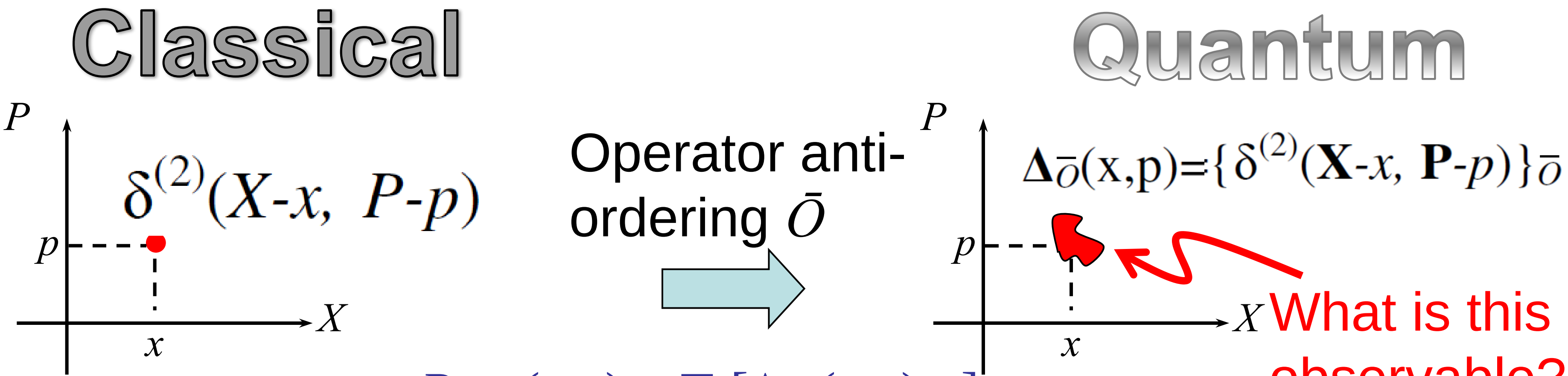
- Using $\mathbf{X} = (\mathbf{a} + \mathbf{a}^\dagger)/\sqrt{2}$, $\mathbf{P} = i(\mathbf{a} - \mathbf{a}^\dagger)/\sqrt{2}$
  $\rightarrow \alpha = x + ip$

1. Expand the density matrix in a particular ordering O
2. Put $\mathbf{a} \rightarrow \alpha$ and $\mathbf{a}^\dagger \rightarrow \alpha^*$
3. The result is the O ordered quasi-prob. Distribution, $Pq_O(x,p)$

Quasi-Prob. Function, Pq_O	Ordering, O	Ordering Definition
Q	Anti-normal, AN	$\mathbf{a}$ to the left of $\mathbf{a}^\dagger$
Wigner	Symmetric ,W	evenly waited sum of all the orderings of $\mathbf{a}^\dagger$ and $\mathbf{a}$
P	Normal, N	$\mathbf{a}^\dagger$ to the left of $\mathbf{a}$

Direct Measurements of Quasi Probability distributions

- For an O ordered distribution measurements are anti-ordered, $\bar{O}$
- Classical measurement is a Dirac delta, rastered over all x and p



$$Pq_O(x,p) = \mathrm{Tr}[\Delta_{\bar{O}}(x,p)\,\boldsymbol{\rho}]$$

Quasi-Prob, Pq_O	Ordering O	Dirac Delta, $\Delta_{\bar{O}}(x,p)$	Experiments & Theory
Q	Normal, N	$\Delta_{AN}(x,p) = \lvert\alpha\rangle\langle\alpha\rvert$	Shapiro, Yuen
Wigner	Symmetric, W	$\Delta_W(x,p)= \mathbf{\Pi}(x,p)$ parity about (x,p)	Banaszek, Haroche, Silberhorn, Smith
P	Anti-N,AN	$\Delta_N(x,p)\neq$observable	

G. S. Agarwal and E. Wolf, *Phys. Rev. D*,**2** (1970) pp. 2161–2186.

X-P ordered Quasi-Prob Distributions

- Two more orderings:

Standard S: **X** to the left of **P**

Anti-Standard AS: **P** to the left of **X**

For the Standard ordering, following our quantization procedure the corresponding Quasi-Probability distribution is:

$$\mathrm{Pq}_S\,(x,p) = \mathrm{Tr}[\boldsymbol{\Delta}_{AS}(x,p)\,\boldsymbol{\rho}]$$

$$\begin{aligned}\boldsymbol{\Delta}_{AS}(x,p) &= \{\delta^{(2)}(\mathbf{X}\text{-}x,\ \mathbf{P}\text{-}p)\}_S \\ &= \delta(\mathbf{P}\text{-}p)\delta(\mathbf{X}\text{-}x,) \\ &= |\mathrm{p}\rangle\langle\mathrm{p}||\mathrm{x}\rangle\langle\mathrm{x}|\end{aligned}$$

$$\mathrm{Pq}_S\,(x,p) = \mathrm{Tr}[|\mathrm{p}\rangle\langle\mathrm{p}||\mathrm{x}\rangle\langle\mathrm{x}|\,\rho] = \langle\mathrm{p}||\mathrm{x}\rangle\langle\mathrm{x}|\,\rho\,|\mathrm{p}\rangle = \mathrm{D}_\rho(\mathrm{x,p})$$

1. The standard ordered distribution is the Dirac distribution!
2. Expectation values = overlap integral, $\langle \mathbf{B}\rangle = \int \mathrm{Pq}_{AS} \cdot \mathrm{Pq}_S\ dxdp$
3. Marginals are equal to Prob(x) and Prob(p)

Bayes' Law and Weak Measurement

A. M. Steinberg, Phys. Rev. A, 52, 32 (1995):
Weakly measured probabilities (e.g. Dirac Dist.) satisfy Bayes' Law.

H. F. Hofmann, New Journal of Physics, 14, 043031 (2012):
Use Baye's law to propagate the Dirac Distribution (like in classical physics!)

1. Generalize Dirac Distribution (no longer anti-standard ordered):

$$\mathrm{Pq}_D(x,q,k,p) = \langle \delta(\mathbf{P}-p)\delta(\mathbf{K}-k)\delta(\mathbf{Q}-q)\delta(\mathbf{X}-x)\rangle$$

2. Use Baye's Law to propagate the Dirac Dist:

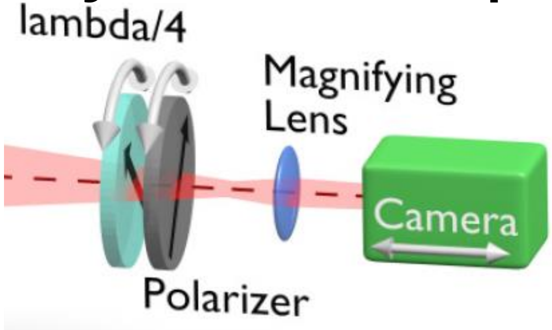
$$\begin{aligned}\mathrm{Pq}_{AS}(x,k) &= \sum_{x,p} \mathrm{Pq}_{\mathrm{D}}(x,q,k,p) \\ &= \sum_{x,p} \mathrm{Pq}_{\mathrm{D}}(q,k|x,p)\cdot \mathrm{Pq}_{AS}(x,p)\end{aligned}$$

3. Use Eq 1 and the formula for the Dirac Dist to find the propagator:

$$\mathrm{Pq}_{\mathrm{D}}(q,k|x,p) = \frac{\mathrm{Pq}_{\mathrm{D}}(x,q,k,p)}{\mathrm{Pq}_{AS}(x,p)} = \frac{\langle p|k\rangle\,\langle k|q\rangle\,\langle q|x\rangle}{\langle p|x\rangle}$$

- The propagator is a weak conditional probability, made up of state overlaps

Bayesian Propagation of the Dirac Distribution



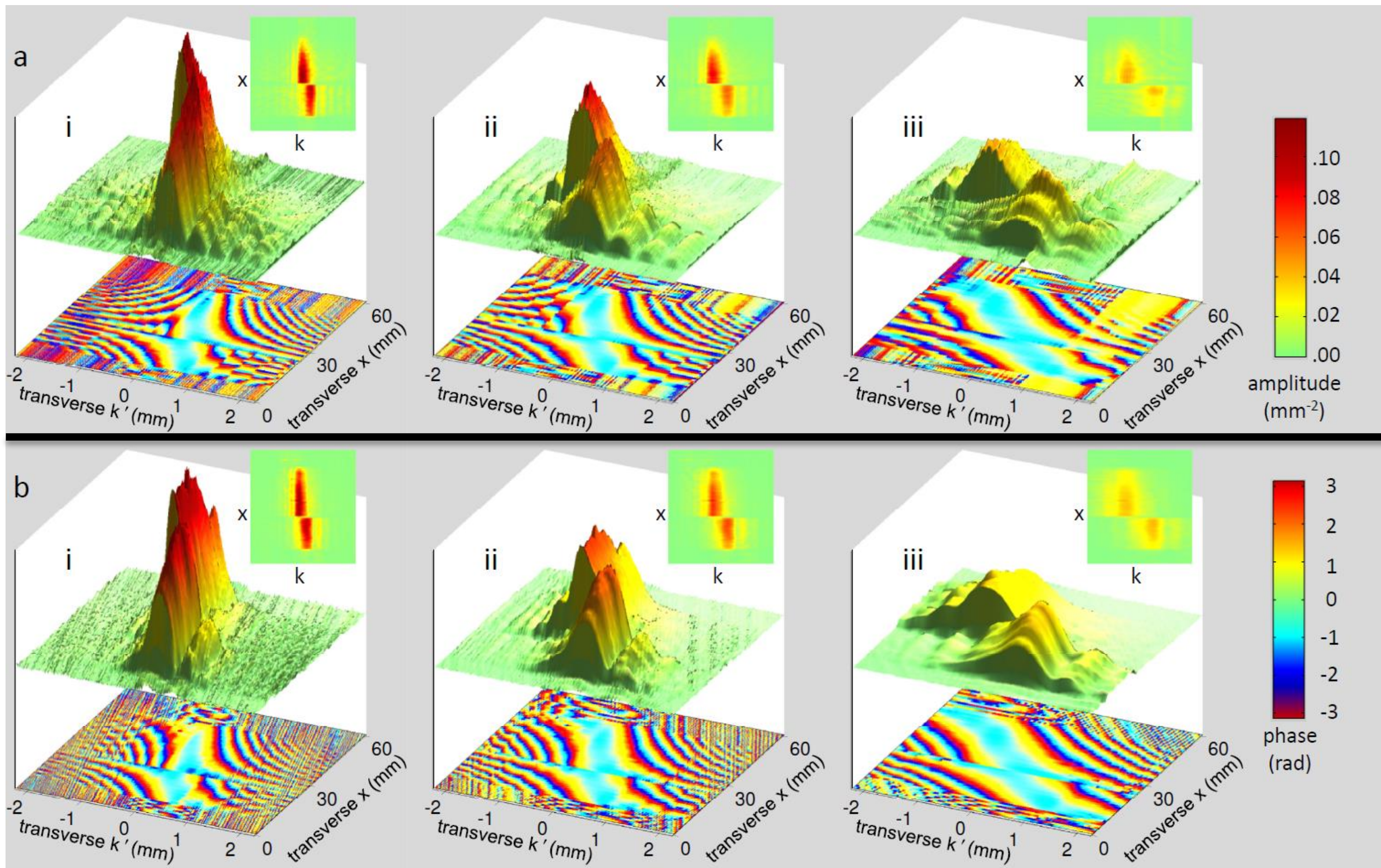
Move camera by Δz

$$D_\rho(x,p) \longrightarrow D_\rho(x, a\cdot p + b\cdot x)$$

Hybrid of variable of x and p, depending on Δz

Experimental Dirac Dist.

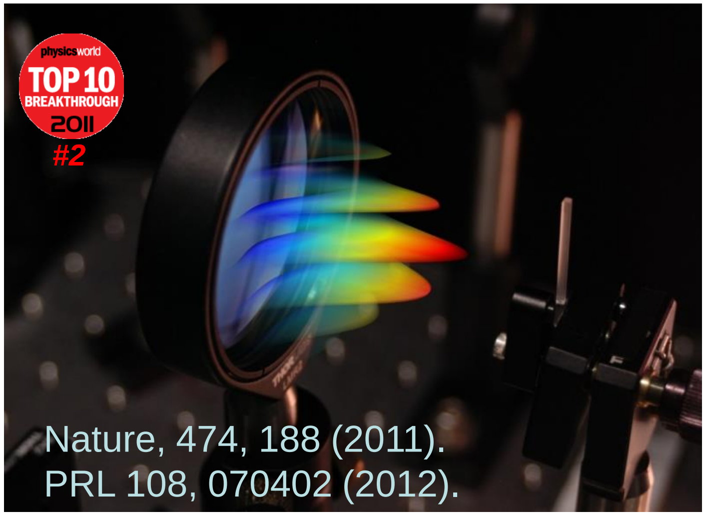
Theoretical Prediction



Δz

Direct measurement of the wavefunction

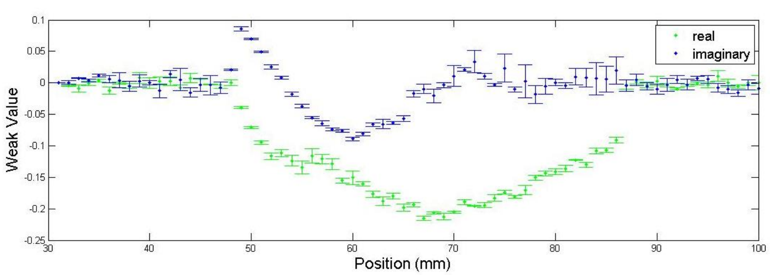
- A slice through the Dirac Distribution D(x,p) is proportional to the quantum wavefunction, e.g. p=0

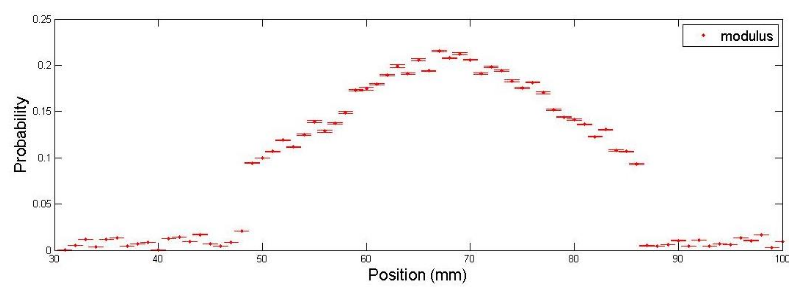


$$D(x,p) = \langle p|x\rangle\langle x|\psi\rangle\langle\psi|p\rangle$$

for p=0,

$$D(x,0) = \langle p{=}0|x\rangle\langle x|\psi\rangle\langle\psi|p{=}0\rangle$$
$$= k\cdot\psi(x)$$





Conclusions

- Like the Wigner function and the Q and P-functions, the Dirac Distribution is an example of an ordered quasi-probability distribution.

- It is directly measured in a particularly straightforward way (weak X then strong P).

- Like a classical x-p distribution, it can be propagated via Baye's Law (see Hofmann)

- The 2nd measurement (e.g. P) can be weak too → in situ state determination!

Who is this quasi-probability distribution?

Kirkwood-Tertselsky-Dirac-
Rihaczek-Marginau-Hill-Walther-
Wolf-Ackroyd-Mehta Distribution