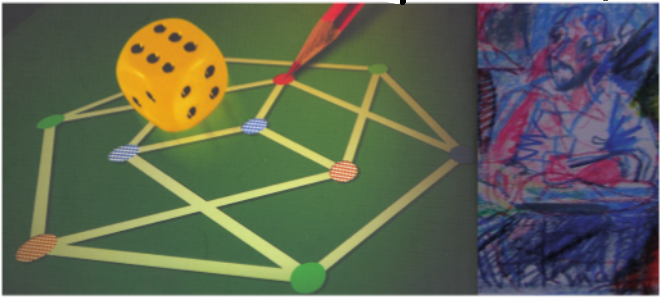
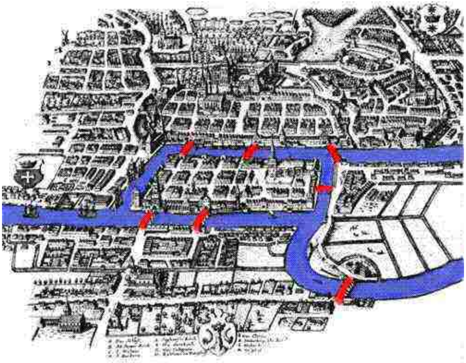
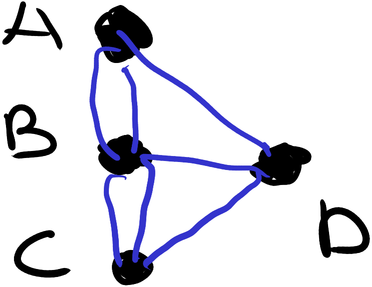
How I learned to do Mathematics

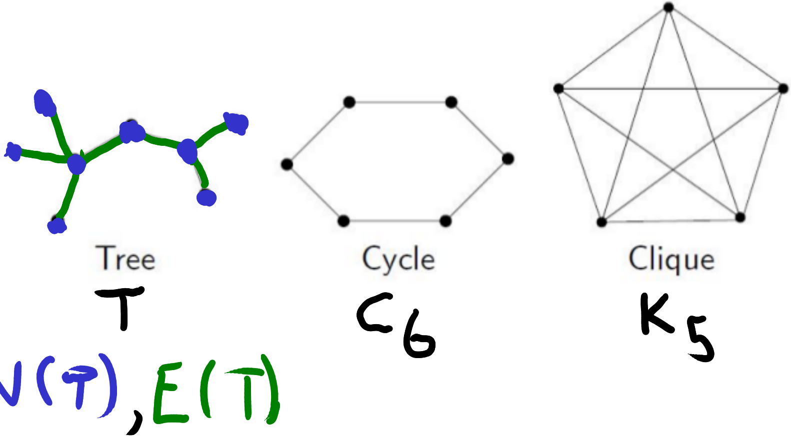


The First(?) Routing Problem

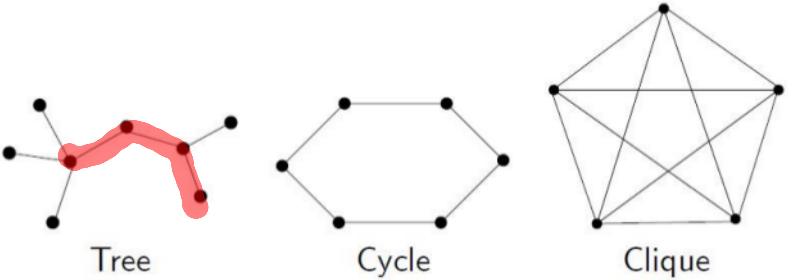




Three Graphs



Graphs and Connectivity



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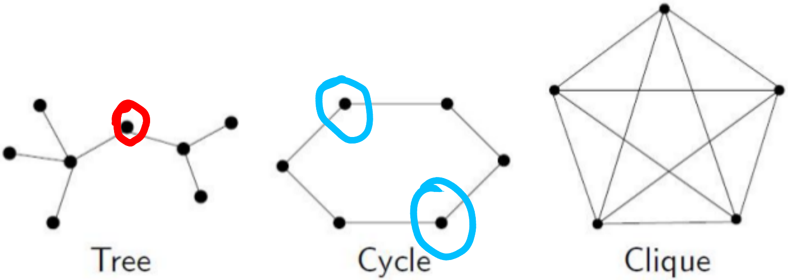

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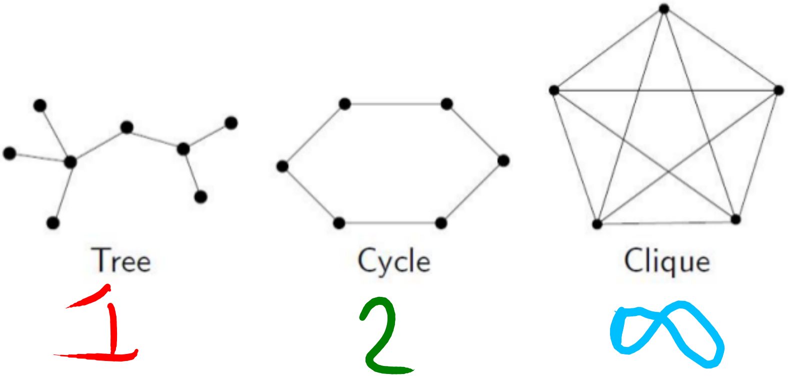
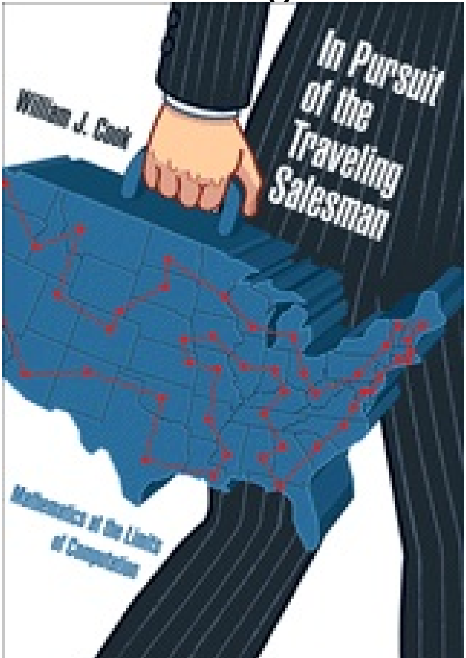
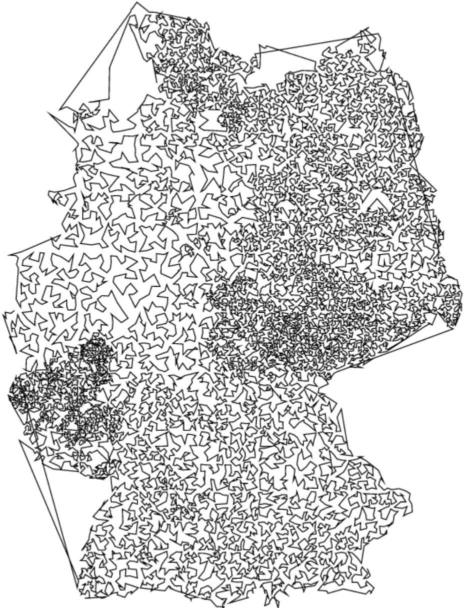


The Travelling Salesman Problem





Conflict Graphs, Stable Sets, and Colouring

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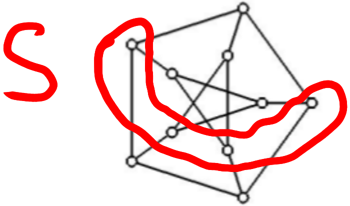

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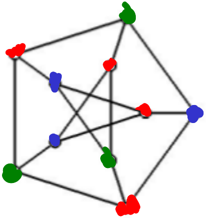

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$\chi(G)$,the chromatic number of G, is the minimum number of stable sets in a partition of $V(G)$.

Handling Large Graphs: An Enduring Problem

As far as the problem of the seven bridges of Konigsberg is concerned, it can be solved by making an exhaustive list of all possible routes, and then determining whether or not any route satisfies the conditions of the problem. Because of the number of possibilities, this method of solution would be too difficult and laborious, and in other problems with more bridges it would be impossible.

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Euler, 1736 (in Latin)

A Framework: Computational Complexity

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P v. NP-complete

A Technique: Polyhedral Combinatorics

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The Colouring ILP

$\chi(G) = \ min \, \Sigma_{S \in \mathcal{S}(G)} \ x_S$

subject to:

$\forall v \in V : \quad \Sigma_{v \in S} \ x_S = 1$

$x \geq 0, x$ integer.

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$\chi^f(C_5) = 2.5.$

A Second Technique:
Global Results via Local Analysis

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Structural Decomposition

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Global Results via Local Analysis

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The Probabillistic Method

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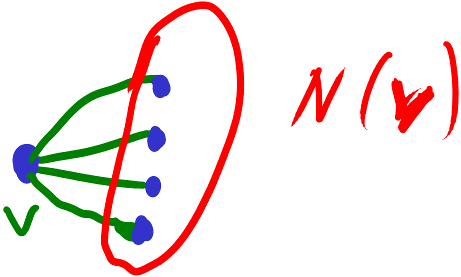

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- the degree of v, denoted $\delta(v)$, is $|N(v)|$
- Δ is the maximum degree of a vertex in G
- $\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$

A Conjecture

$$\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$$

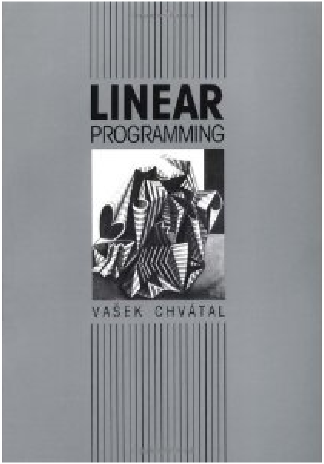
Lessons from Vasek I

Look for what Hilbert calls

the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives

Lessons from Vasek II

Write, and then rewrite, and rewrite and rewrite and rewrite until you get it right

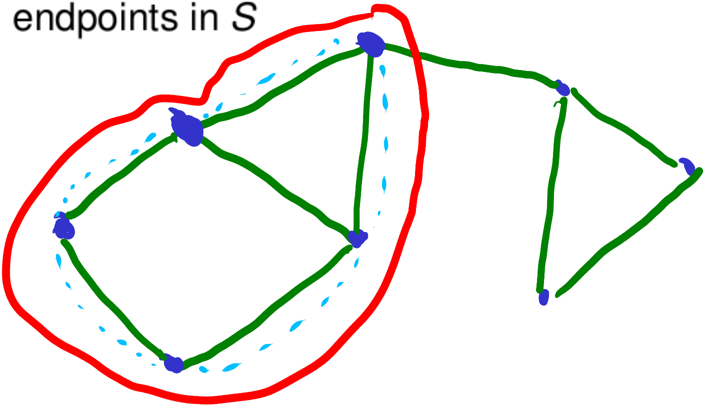


lp

Perfect Graphs

Perfect Graphs

- For $S \subseteq V(G)$, the subgraph $G[S]$ induced by S has vertex set S and contains all the edges of G with both endpoints in S



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- A graph G is perfect if each of its induced subgaphs H satisfies $\chi(H) = \omega(H)$
-

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Given an optimal fractional colouring, rip out a colour class and recurse.

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Can find a fractonal colouring of a perfect graph in polynomial time (Grotschel, Lovasz, & Schrijver, 1979).

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- SPGC(Berge 1961): If G is Berge, it is perfect.
- or equivalently: a graph is minimally imperfect precisely if it is C_{2k+1} or $\overline{C_{2k+1}}$ for some $k \geq 2$.

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G is perfect precisely if $\overline{G}$ is. (Lovasz 1972)

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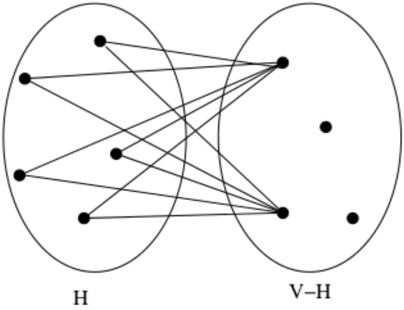

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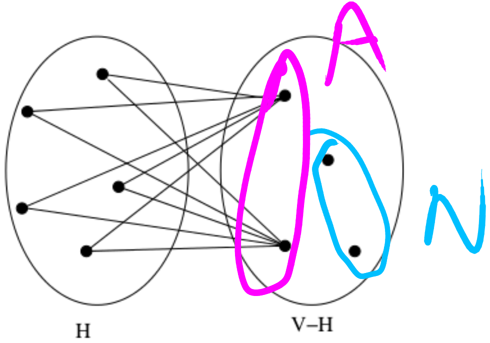
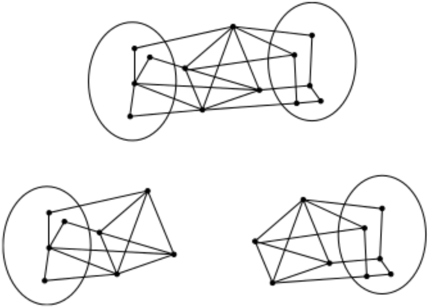


Clique Cutsets



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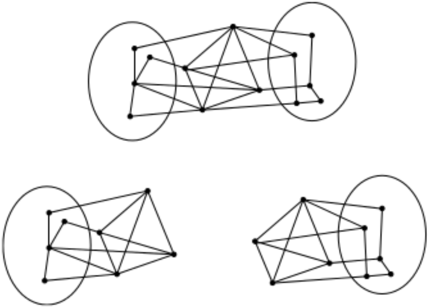

No minimal imperfect graph has a clique cutset.

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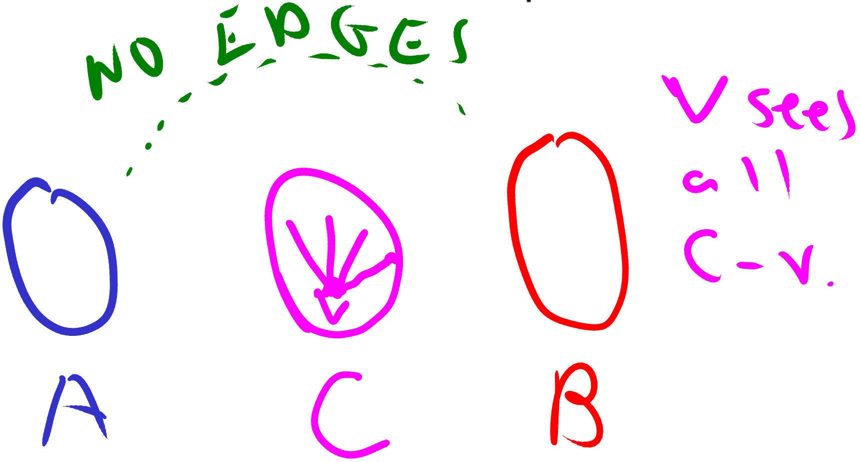
Corollary: Every triangulated graph is perfect.

Star Cutsets and Perfect Graphs



Theorem: No minimal imperfect graph has a star cutset (Chvatal 1985)

Star Cutsets and Strongly Berge Graphs

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Corollary: Every Strongly Berge Graph is perfect.

Skew Cutsets and Even Pairs

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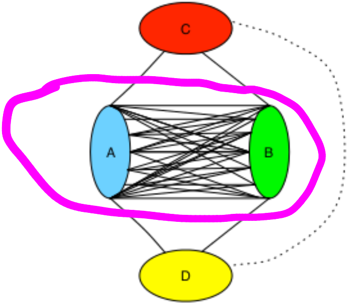

cutset C, $\overline{G[C]}$ is disconnected.

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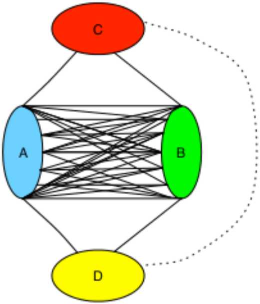

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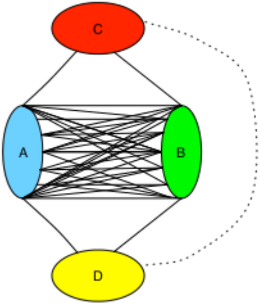

Conjecture: No minimal imperfect graph has a skew cutset (Chvatal 1985)

Theorem: No minimal imperfect graph has an even pair(Meyniel 1987) every induced x-y path P has |E(P)| even

My introduction to Minors and Models

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B. A. Reed

Graph Minors I:

Rooted Routing

July 10, 2007

Springer
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Contents

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If G contains no K_l model then it has an $l-1$ colouring.

A Fractional Hadwiger's Conjecture

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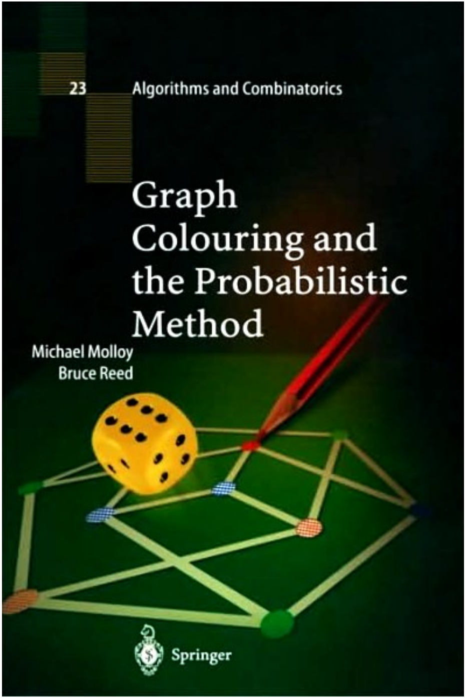
Theorem: If G has no K_l model then $\chi^f(G) \leq 2l - 2$ (R. & Seymour, 1998).


23
Algorithms and Combinatorics
Graph Colouring and the Probabilistic Method
Michael Molloy
Bruce Reed
Springer

A Global/Local Lemma

If $\mathcal{A}$ is a family of events satisfying:

$$\sum_{E \in \mathcal{A}} \mathrm{Prob}(E) < 1$$

then with positive probability none of the (bad) events in $\mathcal{A}$ occurs.

Bounding χ using χ^f

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Finding Nearly Optimal Colourings

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1. A Local Local Lenma

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2. Bells and Whistles

The Lovasz Local Lemma

If $\mathcal{A}$ is a family of events satisfying:

for each F in $\mathcal{A}$ there exists $\mathcal{S}(F)$ s.t. F is mutually independent of $\mathcal{A} - \mathcal{S}(F)$, and $\sum_{E \in \mathcal{S}(F)} \mathrm{Prob}(E) < 1/4$

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Bells and Whistles

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1. Special Probability Distributions

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Some Results

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3. Determining The Threshold k_Δ for which $\chi > \Delta - k_\Delta$ is a local property in graphs of maximum degree Δ (Molloy & R. 2001/in press).

Conclusion via An Alternative Title

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Some Thoughts on Writing A Thesis