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Weakly non-linear completely resonant hamiltonian PDEs and the problem of weak turbulence

(Toronto, 10 January, 2014)

1 Introduction: weak turbulence (WT)

(One of) origines: Rudolf Peierls, in *Annalen der Physik* **3** (1929). See in Selected Papers of Sir Rudolf Peierl, World Scientific, 1997. Modern state of affairs see in

[ZLF] Zakharov, Lvov, Falkovich, *Kolmogorov Spectra of Turbulence*, Springer 1992.

[Naz] S. Nazarenko, *Wave Turbulence*, Springer 2011.

The method of WT applies to various equations. E.g., to NLS:

A) *Deterministic setting.* Consider NLS equation:

$$\dot{u} - i\Delta u + i\rho|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d); \quad \rho = \text{ const}, \;\; \rho > 0.$$

WT deals with small solutions $u \sim \varepsilon,\ 0 < \varepsilon \ll 1$. Let us better consider solutions of order 1 of the rescaled equation

(NLS) $$\dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L\,.$$

Take the exponential basis $\{e_{\mathbf{k}} = e^{i\mathbf{k}\cdot x}, \mathbf{k} \in \mathbb{Z}^d/L =: \mathbb{Z}^d_L\}$. Then

$$-\Delta e_{\mathbf{k}} = \lambda_{\mathbf{k}} e_{\mathbf{k}}; \quad \lambda_{\mathbf{k}} = |\mathbf{k}|^2, \quad \mathbf{k} \in \mathbb{Z}^d_L\,.$$

So there is plenty of exact resonances in the spectrum of the linear system, corresponding to $\varepsilon = 0$. – This is a prerequisite for WT.

Now we have an extreme case: the linear system is *completely resonant* – all its solutions are periodic with the same period $2\pi L^2$. THIS is not needed for the WT, but I will use this property in my analysis of the equation.

Decompose u in the Fourier series, $u(t,x)=\sum u_{\mathbf{k}}(t)e_{\mathbf{k}}(x)$, and write (NLS) as

(*) $$\dot u_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar u_{\mathbf{k}_3}\,, \qquad \mathbf{k}\in\mathbb{Z}^d_L.$$

In WT they do the following:

◇ Study solutions for $(*)$ with a "typical" initial data $u(0)=u^0$, during "long" time.

Claim: *For large values of time only resonant terms in* $(*)$ *are important.*

◇ Study solutions when $\varepsilon\to 0,\ L\to\infty$, by replacing "everywhere" sums $\sum_{\mathbf{k}\in\mathbb{Z}^d_L}$ by integrals $\int_{\mathbf{k}\in\mathbb{R}^d}$. In particular, study under that limit the *energy spectrum* $|u_{\mathbf{k}}(t)|^2$, and prove that it has the *Kolmogorov-Zakharov form*:

(KZ spectrum) $$\langle |u_{\mathbf{k}}(t)|^2\rangle \sim |\mathbf{k}|^{-\varkappa}, \qquad \varkappa>0\,,$$

if $|\mathbf{k}|$ "belongs to the inertial range". Here "$\langle\cdot\rangle$" indicates certain averaging.

It is not quite clear in what order we send $\varepsilon\to 0$ and $L\to\infty$. It may be better to talk not about the limit of WT, but about WT limits.

“... the correct ordering of the limiting processes is obscure.”

(Benney and Saffman ”Nonlinear Interactions of Random Waves in a Dispersive Medium” Proc. Royal Soc. A (1966))

B) *Stochastic setting.* Following

V. Zakharov, V.L'vov, in *Radiophys. Quant. Electronics (1975)*,

also see in

Cardy, Falcovich, Gawedzki "Non-Equilibrium Stat. Phys. and Turbulence", CUP 2008.

consider small solutions of NLS equation with small damping and small random force:

(ZL) $\quad \dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = -\nu(-\Delta + 1)^p u + \sqrt{\nu}\, \langle \text{rand. force} \rangle, \quad x \in \mathbb{T}^d_L,$

where $\varepsilon, \nu \ll 1$. Here ν – inverse time-scale of the forced oscillations; ε – amplitude of small oscillations. Some relation between ν and ε are imposed.

Random Force is

$$\sum b_{\mathbf{k}} \frac{d}{dt} \boldsymbol{\beta}^{\mathbf{k}}(t) e^{i\mathbf{k}\cdot x}, \quad b_{\mathbf{k}} > 0 \text{ and } b_{\mathbf{k}} \to 0 \text{ fast}, \quad \mathbf{k} \in \mathbb{Z}^d_L\,,$$

where $\{\boldsymbol{\beta}^{\mathbf{k}}(t)\}$ – independent standard complex Wiener processes.

Fact: As $t \to \infty$, solution of (ZL) converges in distribution to a stationary measure $\mu_{\varepsilon,\nu}$ of the equation (which is a "statistical equilibrium of the equation"):

$$\mathcal{D}u(t) \rightharpoonup \mu_{\varepsilon,\nu} \qquad \text{as } t \to \infty.$$

Similar to the deterministic case, Zakharov - L'vov do the following:

◇ Write the equation in Fourier:

$$\dot{u}_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho\, i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} - \nu(\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + \sqrt{\nu}\, b_{\mathbf{k}} \dot{\boldsymbol{\beta}}^{\mathbf{k}}(t)$$

The term $i\rho \sum u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3}$ is hamiltonian, with the Hamiltonian

$$\mathcal{H}^4 = \frac{\rho}{4} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} \bar{u}_{\mathbf{k}_4}.$$

◇ They build a small parameter from ε, ν, L and study solutions when $\langle$small parameter$\rangle \to 0,\ L \to \infty$. The goal is to calculate corresponding (KZ) spectrum.

Same remark as before has to be made concerning the two limits $\langle$small parameter$\rangle \to 0$ and $L \to \infty$: it is unclear in which order they should be taken.

We choose $\varepsilon^2 = \nu$ – this is within the bounds, usually imposed in physics (cf. [Naz]). It is illuminating to pass to the slow time $\tau = \nu t$:

(ZL) $$u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta + 1)^p u + \langle \text{rand. force} \rangle', \quad x \in \mathbb{T}^d_L.$$

This is the equation I will discuss, mostly following my work with Alberto Maiocchi, who is now a post-doc in Paris.

We suggest to study the WT limits (at least, some of them) by splitting the limiting process in two steps:

I) prove that when $\nu \to 0$, main characteristics of solutions u^ν have limits of order one, described by certain well posed *effective equation*.
II) Show that main characteristics of solutions for the effective equation have non-trivial limits of order one, when $L \to \infty$ and $\rho = \rho(L)$ is a suitable function of L.

Step I has been done rigorously, and I discuss it in this talk. I stress that the results of Step I along cannot justify the predictions of WT since the (KZ spectrum) cannot hold when the period L is fixed and finite.
At the end of my talk I will show that a heuristic argument *a-la* WT with a suitable choice of the function $\rho(L)$ allows to justify Step II and leads under the limit $L \to \infty$ to a Kolmogorov-Zakharov type kinetic equation and a (KZ spectrum).

It seems that in physics this is called the *Litvak-Hasselman approach.*

2 Averaging for PDEs without resonances

In my works $[KP1]$ SK $\&$ A.Piatnitski, JMPA (2008); $[K2]$ SK, GAFA (2010); $[K3]$ SK, Ann. Inst. Fourier - PR, 2013.

I studied the long-time behaviour of solutions for perturbed hamiltonian PDE without strong resonances, for $L = 1$. Namely, in [KP1,K2] I considered equations like

$$\dot{u} - iu_{xx} + i|u|^2 u = \nu(u_{xx} - u) + \sqrt{\nu}\,\langle \text{rand. force} \rangle, \quad x \in S^1,$$

and in [K3] – equations like

$$(*) \quad \dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu(-\Delta + 1)^p u + \sqrt{\nu}\,\langle \text{rand. force} \rangle, \quad x \in \mathbb{T}^d,$$

where $p \in \mathbb{N}$ and $V(x)$ is such that there are no resonances in the spectrum of $-\Delta + V(x)$. The key idea was suggested in [K2] – describe the long-time behaviour of the actions in the perturbed equations, using certain auxiliary Effective Equation. This is a well posed quasilinear SPDE with a non-local nonlinearity. For eq. $(*)$ without resonances, the Effective Equation is linear and does not depend on the Hamiltonian term $\nu i|u|^2 u$.

3 Averaging for PDEs with resonances

Now the new results.

[KM1] SK & A. Maiocchi, Preprint, arXiv 1311.6793

[KM2] SK & A. Maiocchi, Preprint, arXiv 1311.6794.

We apply the method of [KP1-K3] to the equation of Zakharov-L'vov with $\varepsilon^2 = \nu$, written using the slow time $\tau = \nu t$:

(ZL) $$u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta + 1)^p u + \langle \text{rand. force} \rangle',$$

We write $u(\tau, x) = \sum_{\mathbf{k}\in\mathbb{Z}^d_L} u_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot x}$, and re-write the equation in Fourier:

$$\frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

where $\mathbf{k} \in \mathbb{Z}^d_L$. We wish to control the asymptotic behaviour of the actions $\frac{1}{2}|u_{\mathbf{k}}|^2(\tau)$ and other characteristics of solutions via suitable effective equation. The Effective Equation for (ZL) may be derived through the *interaction representation*, i.e. by transition to the fast rotating variables a:

$$a_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} v_{\mathbf{k}}(\tau), \qquad \mathbf{k} \in \mathbb{Z}_L^d$$

(*the variation of constant*). Note that

(*) $$|a_{\mathbf{k}}(\tau)| \equiv |v_{\mathbf{k}}(\tau)|.$$

In these variables the (ZL) equation reeds

$$\frac{d}{d\tau} a_{\mathbf{k}} = -(\lambda_{\mathbf{k}} + 1)^p a_{\mathbf{k}} + b_{\mathbf{k}}\, e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} \frac{d}{d\tau} \beta^{\mathbf{k}}(\tau)$$
$$- i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \bar{a}_{\mathbf{k}_3} \exp\left(-i\nu^{-1}\tau(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2} - \lambda_{\mathbf{k}_3} - \lambda_{\mathbf{k}})\right) .$$

The terms, constituting the nonlinearity, oscillate fast as ν goes to zero, unless the sum of the eigenvalues in the second line vanishes. So only the terms for which this sum equals zero contribute to the limiting dynamics. The processes $\{\tilde{\boldsymbol{\beta}}^{\mathbf{k}}(\tau),\ \mathbf{k} \in \mathbb{Z}_L^d\}$ such that $\frac{d}{d\tau}\tilde{\boldsymbol{\beta}}^{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} \frac{d}{d\tau}\beta^{\mathbf{k}}(\tau)$ also are stand. independent complex Wiener processes. Accordingly, the effective equation should be the following damped/driven hamiltonian system

(Eff.Eq.) $$\frac{d}{d\tau} v_{\mathbf{k}} = -(\lambda_{\mathbf{k}} + 1)^p v_{\mathbf{k}} - R_{\mathbf{k}}(v) + b_{\mathbf{k}} \frac{d}{d\tau} \tilde{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d ,$$

where $R_{\mathbf{k}}(v)$ is the resonant part of the hamiltonian nonlinearity:

$$R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k} \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3} .$$

It is easy to see that $R(v)$ is the hamiltonian vector field $R = i\nabla\mathcal{H}^4_{\text{res}}$, where $\mathcal{H}^4_{\text{res}}$ is the resonant part of the Hamiltonian $\mathcal{H}^4$:

$$\mathcal{H}^4_{\text{res}} = \frac{\rho}{4} \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4 \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}_4|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3} \bar{v}_{\mathbf{k}_4} .$$

◇ We have to impose some restrictions on p and d to make (ZL) well posed. E.g., $p = 1, \quad d \leq 3$ (if $p > 1$, then d may be bigger than 3).

Properties of $\mathcal{H}^4_{\text{res}}$ and of Eff. Eq.:

Lemma. 1) $\mathcal{H}^4_{\text{res}}$ has two convex quadratic integrals of motion, $H_0 = \sum |v_{\mathbf{k}}|^2$ and $H_1 = \sum(|v_{\mathbf{k}}|^2|\mathbf{k}|^2)$.

2) The hamiltonian vector-field $i\nabla\mathcal{H}^4_{\text{res}}(v)$ is Lipschitz in sufficiently smooth Sobolev spaces.

3) (EffEq) is well posed in sufficiently smooth Sobolev spaces.

- So (Eff. Eq.) is similar to the 2d NSE on a torus!

(ZL)

$$\frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

$$(EffEq) \qquad \frac{d}{d\tau}v_{\mathbf{k}} = -R_{\mathbf{k}}(v) - (\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}^d_L.$$

Actions of a solution $u^\nu(\tau)$ are

$$I^\nu_{\mathbf{k}}(\tau) = \tfrac{1}{2}|u^\nu_{\mathbf{k}}(\tau)|^2, \qquad \mathbf{k} \in \mathbb{Z}^d_L.$$

Theorem 1. Let $\{u^\nu_{\mathbf{k}}(\tau)\}$ and $\{v_{\mathbf{k}}(\tau)\}$ be solutions of (ZL) and (EffEq) with same initial data. Then, for each $\mathbf{k}$ and for $0 \le \tau \le 1$,

$$\mathcal{D}I^\nu_{\mathbf{k}}(\tau) \rightharpoonup \mathcal{D}\tfrac{1}{2}|v_{\mathbf{k}}(\tau)|^2 \text{ as } \nu \to 0.$$

Does the effective equation control the angles $\varphi_{\mathbf{k}} = \arg u_{\mathbf{k}} =: \varphi(u_{\mathbf{k}})$? No, instead it controls the angles of of the a-variables, $a^\nu_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}v^\nu_{\mathbf{k}}(\tau)$, which fast rotate compare to the angles $\varphi_{\mathbf{k}}$.

Now consider a stationary measure μ^ν for (ZL). Let $u_{st}^\nu(\tau) = (u_{st\,\mathbf{k}}^\nu(\tau), \mathbf{k} \in \mathbb{Z}_L^d)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}(u_{st}^\nu(\tau)) \equiv \mu^\nu.$$

Theorem 2. Let Eff. Eq. has a unique stationary measure m_0. Then

$$\mu^\nu \rightharpoonup m_0 \quad \text{as} \quad \nu \to 0.$$

So, if in addition the (ZL) equation has a unique stationary measure, then for ANY its solution $u^\nu(\tau)$ we have

$$\lim_{\nu\to 0} \lim_{\tau\to\infty} \mathcal{D}\big(u^\nu(\tau)\big) = m_0.$$

But when Eff. Eq. has a unique stat. measure?

Theorem 3. 1) Let $p \geq 1$. Then Eff. Eq. has a unique stationary measure if $d \leq 3$.

2) Take any d. Then Eff. Eq. has a unique stationary measure if $p \geq p_d$ for a suitable $p_d \geq 0$.

4 Limit $L \to \infty$ for the Eff. Eq. (on the physical level of accuracy).

Since Eff. Eq. is like 2d NSE on $\mathbb{T}^2$, then this is like the limit “period to infinity” for 2d NSE, which is well known to be complicated. But Eff. Eq. is simpler!

Considre the Eff. Eq. :

$$\frac{d}{d\tau} v_{\mathbf{k}} = -R_{\mathbf{k}}(v) - \gamma_{\mathbf{k}} v_{\mathbf{k}} + b_{\mathbf{k}} \frac{d}{d\tau} \boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d,$$

$$(EffEq) \qquad R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k} \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3}.$$

Here $\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$.

Consider the moments

$$M^{\mathbf{k}_1,\dots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1},\dots\mathbf{k}_{n_1+n_2}}(\tau) = \mathbf{E}(v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{n_1}} \bar{v}_{\mathbf{k}_{n_1+1}} \dots \bar{v}_{\mathbf{k}_{n_1+n_2}}).$$

Physical Assumptions:

i) *Quasi-Gaussian approximation:*

$$M^{\mathbf{l}_1,\mathbf{l}_2}_{\mathbf{l}_3,\mathbf{l}_4} \sim M^{\mathbf{l}_1}_{\mathbf{l}_1} M^{\mathbf{l}_2}_{\mathbf{l}_2} (\delta^{\mathbf{l}_3}_{\mathbf{l}_1} + \delta^{\mathbf{l}_4}_{\mathbf{l}_1})(\delta^{\mathbf{l}_3}_{\mathbf{l}_2} + \delta^{\mathbf{l}_4}_{\mathbf{l}_2}) \, ,$$

and similar for higher order moments.

ii) *Quasi stationary approximation for equations in the chain of moment equations.*

Denote

$$n_{\mathbf{k}} = L^d M^{\mathbf{k}}_{\mathbf{k}}/2 \, , \qquad \tilde{b}_{\mathbf{k}} = L^{d/2} b_{\mathbf{k}}.$$

$n_{\mathbf{k}}$ - normalised energy of the wave-vector $\mathbf{k}$; see in

[ZLF] Zakharov, Lvov, Falkovich, “Kolmogorov Spectra of Turbulence”, Springer 1992.

Accepting the two hypotheses above we get the KZ kinetic equation:

Theorem 4. When $L \to \infty$ we have

$$
(KZ) \qquad \begin{aligned} &\frac{d}{d\tau} n_{\mathbf{k}} = -2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \tilde{b}_{\mathbf{k}}^2 \\ &\quad +4\,\frac{\rho^2}{L} \int_\Gamma d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3} + \gamma_{\mathbf{k}_4}} \\ &\qquad \times \left(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} \right). \end{aligned}
$$

Here Γ is the resonant surface,

$$
\Gamma = \{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathbb{R}^{3d} : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} + \mathbf{k}_3,\ |\mathbf{k}_1|^2 + |\mathbf{k}_2|^2 = |\mathbf{k}|^2 + |\mathbf{k}_3|^2\},
$$

$\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$, $f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ - some bounded smooth function, constructed in terms of the normal frame to Γ at a point $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$.

Because of the dissipation in the Eff. Eq., our (KZ) equation is “better” then usually the (KZ) equations are: the divisor in the integrand has no zeroes.

The KZ spectra.

I recall that $\gamma_k = a|k|^m + b$. Looking for solutions of (KZ), where $n_{\mathbf{k}}$ depends only on $|\mathbf{k}|$ and arguing *a-la* Zakharov, we find that the equation has the the following time-independent homogeneous solutions:

i) if $0 < a \ll b \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-d+2/3}, \quad \text{or} \quad n_{\mathbf{k}} \sim |\mathbf{k}|^{-d}.$$

ii) if $0 < b \ll a \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d-2}{3}}, \quad \text{or} \quad n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d}{3}}.$$

Many happy returns, Walter!