Three-wave resonant interactions: Part 2

Ruth Martin

Department of Applied Math
University of Colorado at Boulder
Advised by Harvey Segur

11 January, 2014

Introduction

$$\frac{\partial A_m}{\partial \tau} + \mathbf{c}_m \cdot \nabla A_m = i\gamma_m A_k^* A_\ell^*.$$

- From the previous talk, we know that the solutions to the three-wave equations have a formal series expansion with five free functions of x.
- We want to show that this expansion is actually meaningful.
- To that end, we look for a radius of convergence for the Laurent series solution to the three-wave PDEs.
- We use what we know about the convergence of the series solution of the ODEs in order to find a radius of convergence for the series solution of the PDEs.

The three-wave equations

- Let $A_m = -\frac{ia_m}{\sqrt{|\gamma_k \gamma_\ell|}}$ and restrict our attention to **one spatial dimension** so that the three-wave equations become

$$\boxed{\frac{\partial a_m}{\partial \tau} + c_m \frac{\partial a_m}{\partial x} = \sigma_m a_k^* a_\ell^*,}$$

with $\sigma_m = \text{sign}(\gamma_m)$.

The three-wave ODEs

- Without spatial dependence, we have

$$\boxed{\frac{da_m}{d\tau} = \sigma_m a_k^* a_\ell^*.} \tag{1}$$

- There are three associated conserved quantities:

$$\begin{aligned} -iH &= a_1 a_2 a_3 - a_1^* a_2^* a_3^*, \\ K_2 &= \sigma_1 |a_1|^2 - \sigma_2 |a_2|^2, \\ K_3 &= \sigma_1 |a_1|^2 - \sigma_3 |a_3|^2, \end{aligned}$$

  where H, K_2, and K_3 are real constants.
- The ODEs (1) constitute a Hamiltonian system.

The three-wave ODEs

$$\boxed{\frac{da_m}{d\tau} = \sigma_m a_k^* a_\ell^*,}$$

- We can reduce our system of ODEs to a lower dimensional Hamiltonian system.
- Write $a_m(\tau) = |a_m(\tau)|e^{i\varphi_m(\tau)}$, $m = 1, 2, 3$.
- Define:

$$\rho = \sigma_1 |a_1|^2 \quad \text{and} \quad \Phi = \varphi_1 + \varphi_2 + \varphi_3.$$

The Hamiltonian system

- The reduced Hamiltonian system is:

$$
\begin{aligned}
H &= -2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\,\sin\Phi,\\
\frac{d\rho}{d\tau} &= 2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\cos\Phi,\\
\frac{d\Phi}{d\tau} &= -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}}+\sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}}\right.\\
&\qquad\left.+\sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\sin\Phi,
\end{aligned}
$$

with $\sigma=\sigma_1\sigma_2\sigma_3$, $\rho=\sigma_1|a_1|^2$, $\Phi=\varphi_1+\varphi_2+\varphi_3$.

Solution of the Hamiltonian system

- Solve the Hamiltonian system analytically in the complex plane to find the solution in terms of the **Weierstrass elliptic function:**

$$\rho(\tau) = \sigma\,\wp(\tau - k; g_2, g_3) + \frac{K_2 + K_3}{3},$$

$$\Phi(\tau) = 2\arctan\left\{\tan\left(\frac{\Phi_0}{2}\right)\exp\left[\int_0^\tau f(\rho(t))\,dt\right]\right\},$$

where

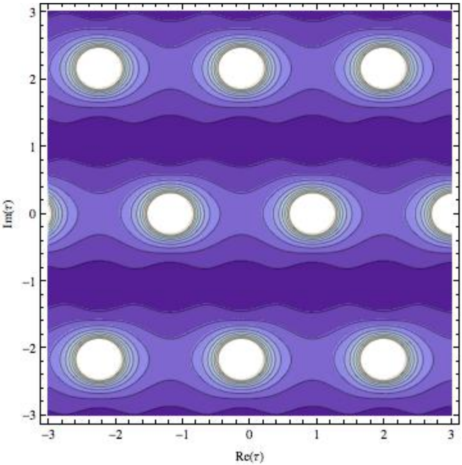
$$g_2 = \frac{4}{3}\left(K_2^2 + K_3^2 - K_2K_3\right),$$

$$g_3 = \frac{4\sigma}{27}(K_2 - 2K_3)(2K_2 - K_3)(K_2 + K_3) + H^2,$$

$$f(\rho) = -\sigma_1\sqrt{\frac{\sigma(\rho - K_2)(\rho - K_3)}{\rho}} - \sigma_2\sqrt{\frac{\sigma\rho(\rho - K_3)}{\rho - K_2}} - \sigma_3\sqrt{\frac{\sigma\rho(\rho - K_2)}{\rho - K_3}}.$$

Example solution

$|\rho(\tau)|$



$\sigma_1 = \sigma_2 = \sigma_3 = 1$

$K_2 = 1,\ K_3 = 2$

$\rho_0 = 2.5$, and $\Phi_0 = \frac{\pi}{3}$

$\omega_1 \approx 1.058,$

$\omega_2 \approx 0.529 + 1.082i$

Laurent series expansion for the ODEs

- The explosive case is well understood.
  - We can determine the general solution to the ODEs in terms of a Laurent series:

$$a_m(\tau) = \frac{e^{i\theta_m}}{\tau_0 - \tau}\Big[1 + \alpha_m(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + \delta_m(\tau_0 - \tau)^3 + \mathcal{O}(\tau_0 - \tau)^4\Big],$$

where

$$\begin{aligned} \alpha_m &= 0, & & \\ \operatorname{Im}(\beta_m) &= 0, & \operatorname{Re}(\beta_1 + \beta_2 + \beta_3) &= 0, \\ \operatorname{Re}(\delta_m) &= 0, & \operatorname{Im}(\delta_1) = \operatorname{Im}(\delta_1) = \operatorname{Im}(\delta_1) &= \delta \end{aligned}$$

Laurent series expansion for the ODEs

$$a_m(\tau) = \frac{e^{i\theta_m}}{\tau_0 - \tau}\left[1 + \beta_m(\tau_0-\tau)^2 + \delta(\tau_0-\tau)^3 + \mathcal{O}(\tau_0-\tau)^4\right],$$

$$\beta_1 = \frac{\sigma}{6}\left(K_2 + K_3\right),$$
$$\beta_2 = \frac{\sigma}{6}\left(K_3 - 2K_2\right),$$
$$\beta_3 = \frac{\sigma}{6}\left(K_2 - 2K_3\right),$$
$$\delta = -\frac{i\sigma H}{6}.$$

There are six free, real-valued constants in the general solution: $\{\theta_1, \theta_2, \tau_0, K_1, K_2, H\}$.

The case $K_2 = K_3 = 0$

Let $K_2 = K_3 = 0$. Then

$$a_m(\tau) = \frac{e^{i\theta_m}}{\tau_0 - \tau}\left[1 - \frac{i\sigma H}{6}(\tau_0 - \tau)^3 + \frac{H^2}{252}(\tau_0 - \tau)^6 + \mathcal{O}(\tau_0 - \tau)^9\right]$$
$$= \frac{e^{i\theta_m}}{\xi}\sum_{n=0}^{\infty} A_{3n}\,\xi^{3n},$$

where $\xi = \tau_0 - \tau$, $A_0 = 1$, $A_3 = -i\sigma H/6$, and

$$(3n-1)A_{3n} + 2A_{3n}^* = -\sum_{p=1}^{n-1} A_{3p}^* A_{3(n-p)}^*, \quad \text{for } n \geq 2.$$

Finding the radius of convergence

- We can determine the **radius of convergence** using the Weierstrass solution, since $a_m(\tau) = \sqrt{\sigma\rho(\tau)}e^{i\varphi_m(\tau)}$.
- $R = \min\left\{2\left|\omega_1\right|, 2\left|\omega_2\right|\right\}$
- When $K_2 = K_3 = 0$, we have

$$g_2 = 0, \quad g_3 = H^2, \quad \text{and} \quad \omega_1 = e^{-i\pi/3}\omega_2 = \frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi g_3^{1/6}}.$$

- That is,

$$R = \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\left|H\right|^{1/3}} \approx \frac{3.06}{\left|H\right|^{1/3}}.$$

Finding the radius of convergence

- We can check numerically that this exact value of R agrees with the radius of convergence provided by the ratio test.
- Since $a_m(\tau) = \frac{e^{i\theta_m}}{\xi} \sum_{n=0}^{\infty} A_{3n}\, \xi^{3n}$, the ratio test tells us that the series converges when

$$\lim_{n\to\infty} \left| \frac{A_{3(n+1)}\, \xi^{3(n+1)}}{A_{3n}\, \xi^{3n}} \right| < 1,$$

or

$$|\xi| < \left(\lim_{n\to\infty} \left| \frac{A_{3(n+1)}}{A_{3n}} \right| \right)^{-1/3} \approx \frac{3.06}{|H|^{1/3}} = R.$$

- Note that this implies $\lim_{n\to\infty} \left| \frac{A_{3(n+1)}}{A_{3n}} \right| = 1/R^3$.

Laurent series expansion for the PDEs

- In the explosive case, we look for a general solution to the PDEs in terms of a Laurent series in τ:

$$a_m(x,\tau) = \frac{e^{i\theta_m(x)}}{\tau_0-\tau}\Big\{1 + \left[B_m^{\mathrm{Re}}(x) + iB_m^{\mathrm{Im}}(x)\right](\tau_0-\tau) + \left[C_m^{\mathrm{Re}}(x) + iC_m^{\mathrm{Im}}(x)\right](\tau_0-\tau)^2 + \left[D_m^{\mathrm{Re}}(x) + iD_m^{\mathrm{Im}}(x)\right](\tau_0-\tau)^3 + \cdots\Big\}.$$

- There will be five free functions of x,

$$\left\{\theta_1(x), \theta_2(x), C_1^{\mathrm{Re}}(x), C_2^{\mathrm{Re}}(x), D_1^{\mathrm{Im}}(x)\right\}.$$

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

- We start by restricting our attention to the case where θ_m is constant for $m = 1, 2, 3$, and $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$.
- In this case, we have

$$\begin{aligned} B_m^{\mathrm{Re}}(x) = B_m^{\mathrm{Im}}(x) &= 0, \\ C_m^{\mathrm{Im}}(x) &= 0, \\ D_m^{\mathrm{Re}}(x) &= 0, \qquad D_1^{\mathrm{Im}}(x) = D_2^{\mathrm{Im}}(x) = D_3^{\mathrm{Im}}(x) = \frac{H(x)}{6}, \end{aligned}$$

so that

$$a_m(x,\tau) = \frac{e^{i\theta_m}}{\tau_0 - \tau}\left[1 + \frac{iH(x)}{6}(\tau_0 - \tau)^3 + \mathcal{O}(\tau_0 - \tau)^4\right].$$

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

- In this simple case, with $\xi = \tau_0 - \tau$, we now have

$$a_m(x,\tau) = \frac{e^{i\theta_m}}{\xi} \sum_{n=0}^{\infty} A_n^m(x)\xi^n,$$

where for $m = 1, 2, 3$,

$$A_0^m(x) = 1, \quad A_1^m(x) = A_2^m(x) = 0, \quad A_3^m(x) = \frac{iH(x)}{6},$$

and

$$(n-1)A_n^m + A_n^{k^*} + A_n^{\ell^*} = c_m A_{n-1}^{m'} - \sum_{p=3}^{n-3} A_p^{k^*} A_{n-p}^{\ell^*}, \quad \text{for } n \geq 4.$$

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

$$a_m(x,\tau) = \frac{e^{i\theta_m}}{\xi}\left\{1 + \frac{iH(x)}{6}\xi^3 + \frac{i}{24}\left(2c_m + c_k + c_\ell\right)H'(x)\,\xi^4 \right.$$
$$\left. + \frac{i}{120}\left(3c_m^2 + c_k^2 + c_k c_\ell + c_\ell^2 + 2c_m\left(c_k + c_\ell\right)\right)H''\,\xi^5 + \cdots\right\}$$

- We want to know the radius of convergence for the series expansion of $a_m(x,\tau)$.
- For further simplification, we consider a particular family of functions $H(x)$, for which:

$$\left\|H^{(n)}(x)\right\| = k^n \left\|H\right\|$$

  - Example: $H(x) = B\sin kx$ or $H(x) = B\cos kx$.

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

- Let $c = \max\{|c_1|, |c_2|, |c_3|\}$.

$$|a_m(x,\tau)| \le \frac{1}{|\xi|}\left\{1 + \frac{\|H\|}{6}|\xi|^3 + \frac{\|H\|}{6}ck|\xi|^4 + \frac{\|H\|}{12}(ck)^2|\xi|^5 + \left[\frac{\|H\|^2}{252} + \frac{\|H\|}{36}(ck)^3\right]|\xi|^6 + \cdots\right\}$$

$$= \frac{1}{|\xi|}\left[1 + \sum_{n=3}^{\infty}\sum_{p=1}^{\lfloor n/3\rfloor} q_{n,p}(ck)^{n-3p}|\xi|^n\right],$$

where $q_{n,p}$ are constants.

- EXAMPLE: The coefficient of $|\xi|^9$ is:

$$q_{9,1}(ck)^6 + q_{9,2}(ck)^3 + q_{9,3} = \frac{\|H\|}{4320}(ck)^6 + \frac{\|H\|^2}{189}(ck)^3 + \frac{\|H\|^3}{4536}.$$

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

- It turns out to be easier to sum down diagonals instead of rows. The index p in $q_{n,p}$ then refers to the pth diagonal.

n	$(ck)^0$	$(ck)^1$	$(ck)^2$	$(ck)^3$	$(ck)^4$	$(ck)^5$	$(ck)^6$	$(ck)^7$	$(ck)^8$
3	$\frac{\|H\|}{6}$	0	0	0	0	0	0	0	0
4	0	$\frac{\|H\|}{6}$	0	0	0	0	0	0	0
5	0	0	$\frac{\|H\|}{12}$	0	0	0	0	0	0
6	$\frac{\|H\|^2}{252}$	0	0	$\frac{\|H\|}{36}$	0	0	0	0	0
7	0	$\frac{\|H\|^2}{126}$	0	0	$\frac{\|H\|}{144}$	0	0	0	0
8	0	0	$\frac{\|H\|^2}{126}$	0	0	$\frac{\|H\|}{720}$	0	0	0
9	$\frac{\|H\|^3}{4536}$	0	0	$\frac{\|H\|^2}{189}$	0	0	$\frac{\|H\|}{4320}$	0	0
10	0	$\frac{\|H\|^3}{1512}$	0	0	$\frac{\|H\|^2}{378}$	0	0	$\frac{\|H\|}{30\,240}$	0
11	0	0	$\frac{\|H\|^3}{1008}$	0	0	$\frac{\|H\|^2}{945}$	0	0	$\frac{\|H\|}{241\,920}$
12	$\frac{11\|H\|^4}{2\,476\,656}$	0	0	$\frac{\|H\|^3}{1008}$	0	0	$\frac{\|H\|^2}{2835}$	0	0
13	0	$\frac{11\,\|H\|^4}{619\,164}$	0	0	$\frac{\|H\|^3}{1344}$	0	0	$\frac{2\|H\|^2}{19\,845}$	0

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

- Rewrite the double sum:

$$|a_m(x,\tau)| \le \frac{1}{|\xi|}\left[1 + \sum_{p=1}^{\infty}\sum_{n=3p}^{\infty} q_{n,p}(ck)^{n-3p}|\xi|^n\right]. \qquad (2)$$

- We can prove that

$$q_{n,p} = \frac{p^{n-3p}}{(n-3p)!}\cdot q_{3p,p}, \quad \text{for } n \ge 3p.$$

- The constants $q_{3p,p}$ are the constants found in the first column of the table.

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

The inner sum in (2) has a simple closed form:

$$\begin{aligned}\sum_{n=3p}^{\infty} q_{n,p}(ck)^{n-3p}|\xi|^n &= \sum_{n=3p}^{\infty} \frac{p^{n-3p}}{(n-3p)!} \cdot q_{3p,p} \cdot (ck)^{n-3p}|\xi|^n \\ &= q_{3p,p}\,|\xi|^{3p} \sum_{n=0}^{\infty} \frac{(ck\,p\,|\xi|)^n}{n!} \\ &= q_{3p,p}\,|\xi|^{3p}\, e^{ckp|\xi|}.\end{aligned}$$

The case where $C_1^{\mathrm{Re}}(x) = C_2^{\mathrm{Re}}(x) = 0$

The bound (2) becomes

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\left[1 + \sum_{p=1}^{\infty} q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|}\right] = \frac{1}{|\xi|}\sum_{p=0}^{\infty} q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|},$$

where we defined $q_{0,0} = 1$.

Finding the radius of convergence

- Finally, we apply the ratio test to find the radius of convergence:

$$\lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}\,|\xi|^{3(p+1)}\,e^{ck(p+1)|\xi|}}{q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|}}\right| < 1$$

$$\implies \qquad |\xi|^3\,e^{ck|\xi|}\cdot\lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}}{q_{3p,p}}\right| < 1.$$

- However, the numbers $q_{3p,p}$ are the coefficients in the Laurent series for the ODEs. We know the radius of convergence in that case, so

$$\lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}}{q_{3p,p}}\right| = \lim_{p\to\infty}\left|\frac{A_{3(p+1)}}{A_{3p}}\right| = \frac{1}{R^3} = \frac{(4\pi)^3\,|H|^{1/3}}{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}.$$

Finding the radius of convergence

- We now know

$$\lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}}{q_{3p,p}}\right| = \frac{(4\pi)^3\cdot\|H\|}{2^3\left[\Gamma\left(\frac{1}{3}\right)\right]^9}.$$

- As a result, the Laurent series expansion for the PDEs converges when

$$|\xi|^3\, e^{ck|\xi|}\cdot\frac{(4\pi)^3\cdot\|H\|}{2^3\left[\Gamma\left(\frac{1}{3}\right)\right]^9} < 1,$$

or

$$|\tau_0-\tau|\, e^{\frac{ck|\tau_0-\tau|}{3}} < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\cdot\|H\|^{1/3}} \approx \frac{3.06}{\|H\|^{1/3}}.$$

Comparison with ODEs

- The radius of convergence for the PDEs is smaller than that for the ODEs due to the factor of $e^{\frac{ck|\tau_0-\tau|}{3}}$.
  - ODEs:
$$|\tau_0-\tau| < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\,|H|^{1/3}} \approx \frac{3.06}{|H|^{1/3}}.$$
  - PDEs:
$$|\tau_0-\tau|\, e^{\frac{ck|\tau_0-\tau|}{3}} < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\cdot\|H\|^{1/3}} \approx \frac{3.06}{\|H\|^{1/3}}.$$

PDEs: Another simple case

- An alternative approach is to keep the phases constant, but set $D_1^{\mathrm{Im}}(x) = 0$ and pick $C_1^{\mathrm{Re}}(x)$ and $C_2^{\mathrm{Re}}(x)$ to be nonzero.
  - In the ODE case, this is equivalent to setting $H = 0$, and keeping K_2 and K_3 nonzero.
  - A particularly special case is that corresponding to $K_3 = 2K_2$. Much of the analysis is the similar, though slightly more complicated, to the previous case. We find

$$\text{ODEs:} \qquad |\tau_0 - \tau| < \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{1024^{1/4} \cdot \pi^{1/2} \cdot \|K\|^{1/2}} \approx \frac{2.62}{\|K\|^{1/2}},$$

$$\text{PDEs:} \quad |\tau_0 - \tau|\, e^{\frac{ck|\tau_0 - \tau|}{2}} < \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{1024^{1/4} \cdot \pi^{1/2} \cdot \|K\|^{1/2}} \approx \frac{2.62}{\|K\|^{1/2}}.$$

Results and future problems

- In two cases, we have found that the radius of convergence for the Laurent series solution to the three-wave PDEs is smaller than the radius of convergence for the three-wave ODEs by a known factor.
  - The factor depends only on the largest group velocity (in magnitude) and the rate at which the derivatives of the free functions grow.
- We would like to determine whether this is true in general, for more than the two special cases we discussed.
  - That is, we want to find a more general radius of convergence for the case where $D_1^{\mathrm{Im}}(x)$, $C_1^{\mathrm{Re}}(x)$, and $C_2^{\mathrm{Re}}(x)$ are all nonzero.
  - We would like to determine what happens when we no longer force the phases to be constant.

Results and future problems

- In two cases, we have found that the radius of convergence for the Laurent series solution to the three-wave PDEs is smaller than the radius of convergence for the three-wave ODEs by a known factor.
  - The factor depends only on the largest group velocity (in magnitude) and the rate at which the derivatives of the free functions grow.
- We would like to determine whether this is true in general, for more than the two special cases we discussed.
  - That is, we want to find a more general radius of convergence for the case where $D_1^{\mathrm{Im}}(x)$, $C_1^{\mathrm{Re}}(x)$, and $C_2^{\mathrm{Re}}(x)$ are all nonzero.
  - We would like to determine what happens when we no longer force the phases to be constant.

Results and future problems

- We did not impose any boundary conditions in x when constructing our Laurent series for the PDEs. However, we could impose boundary conditions on the free functions of x in the Laurent series. This should allow our representation of the solution to be compatible with many types of boundary conditions.
- Our approach is an alternative to using Inverse Scattering mechanics.
- We still do not know how to specify initial data.
- We do not yet know how to replace τ_0 with an arbitrary function of x.
- Once we include $\tau_0(x)$ in our series, we will have a general solution to the nonlinear PDE within the annulus of convergence.

Thank you for your attention.