Justification of the nonlinear Schödinger equation for two-dimensional gravity driven water waves

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Abstract

Abstract: In 1968 V.E. Zakharov derived the Nonlinear Schrödinger equation as an approximation to the 2D water wave problem in the absence of surface tension in order to describe slow temporal and spatial modulations of a spatially and temporarily oscillating wave packet. I will describe a recent proof that the wave packets in the two-dimensional water wave problem in a canal of finite depth can be accurately approximated by solutions of the Nonlinear Schrödinger equation.

This is joint work with Wolf-Patrick Düll and Guido Schneider of the University of Stuttgart

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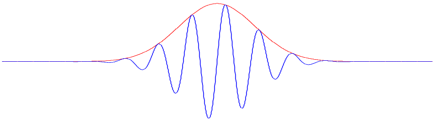
   A. Resonances.

   B. Loss of smoothness.

The NLS approximation

Want to study the evolution of "wave packets" on a fluid surface



The underlying carrier wave (blue) will propagate with the "phase velocity", whereas the envelope (red) will translate with the "group velocity", but as V. Zakharov (1968) argued, the shape of the envelope should evolve on a much slower time scale, and the changes in its shape should be described the the Nonlinear Schrödinger Equation (NLS).

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4. There has been a great deal of activity in recent years that focusses on giving rigorous estimates of the accuracy with which these modulation equations approximate the true motion of the system.
5. Much of this activity was motivated by Walter's paper: *An existence theory for water waves and the Boussinesq and Kortweg-de Vries scaling limits*, Comm. PDE's vol. 10, pp. 787-1003 (1993).

NLS again

The NLS approximation has been one of the last of these modulation equations to yield to rigorous analysis.

1. W. Craig, C. Sulem, P.L. Sulem. *Nonlinear modulation of gravity waves: a rigorous approach* (1992)
2. N. Totz, S. Wu. *A rigorous justification of the modulation approximation to the 2D full water wave problem* (2012).
3. W.-P. Düll, G. Schneider, C.E. Wayne. *Justification of the NLS equation for the evolution of gravity driven 2D surface water waves in a canal of finite depth* (2013).

A model problem

Consider the model problem:

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u - \omega^2 u^2$$
$$u = u(x,t),\ x \in \mathbb{R},\ t \in \mathbb{R}$$

Here, ω^2 is a Fourier multiplier operator, defined by its action on Fourier transforms:

$$\omega^2 u = \mathcal{F}^{-1}(k \tanh(k) \hat{u}(k,t)) \ .$$

Similarities with the water wave problem:

1. The same dispersion relation.
2. Quadratic nonlinear term.
3. The Fourier transform of the nonlinear term vanishes at the origin.

Wavetrains

Note that the linear part of the equation has a family of plane waves:

$$u^L(x,t) = e^{i(kx+\omega(k)^2 t)} \ .$$

It is now common to search for slowly varying wave trains of the nonlinear problem of the form:

$$\Psi^{NLS}(x,t) = \epsilon A(\epsilon x, \epsilon^2 t) e^{i(kx+\omega(k)t)} + complex\ conjugate \ .$$

Then a nonrigorous calculation shows that the amplitude function A satisfies

$$\frac{\partial A}{\partial T} = i\nu_1 \frac{\partial^2 A}{\partial X^2} + i\nu_2 A|A|^2 \ .$$

Timescales, etc.

One can see from this last calculation a part of the reason why the NLS approximation is so difficult to justify rigorously.

In terms of the parameter ϵ which describes the amplitude of the solution, one needs to control the equation for times $\sim \mathcal{O}(\epsilon^2)$ - a *very* long time.

For the KdV regime, for example, one needs only control the evolution for times $\sim \mathcal{O}(\epsilon^{3/2})$.

Justifying the approximation

To *rigorously* justify this approximation we write

$$u(x,t) = \Psi^{NLS} + \epsilon^{\beta} R$$

for $\beta \geqslant 2$.

We then insert this expression for u in our original equation and derive the equation for R.

Note that if $R \sim \mathcal{O}(1)$ for $0 \leqslant t \leqslant \epsilon^{-2}$ then the nonlinear Schrödinger approximation correctly describes the behavior of solutions of our original equation.

The remainder

In order to control the evolution of R over the long times we consider we make two initial changes:

- We rewrite the equation as a system of two first order equations.
- We diagonalize the linear part of the equation.

This leads to the following equation for the remainder R:

$$\frac{\partial R}{\partial t} = \Lambda R + 2\epsilon N(\Psi^{NLS}, R) + \epsilon^{\beta} N(R,R) + \epsilon^{-\beta} \mathrm{Res}(\Psi^{NLS})$$

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- In an abuse of notation R is now a two-component vector - it still, however, is the difference between the NLS approximation and a true solution of our original equation.
- Λ is a 2×2, diagonal matrix operator whose diagonal elements (in Fourier transform variables) are

$$\lambda_j(k) = (-i)^{j-1}\omega(k) = (-i)^{j-1}\sqrt{k\tanh(k)},\ j = 1, 2.$$

The remainder

$$\frac{\partial R}{\partial t} = \Lambda R + 2\epsilon N(\Psi^{NLS}, R) + \epsilon^{\beta} N(R, R) + \epsilon^{-\beta} \mathrm{Res}(\Psi^{NLS})$$

- The bilinear function N has the representation (again in Fourier space) of

$$\hat{N}(U, V)(k) = -\omega(k)(0, ((\hat{U})_1 * (\hat{V})_1)(k))^T$$

- $\mathrm{Res}(\Psi^{NLS})$ measures the amount by which Ψ^{NLS} fails to satisfy the original equation at any given time. We can make it as small as we wish but choosing the approximation appropriately. (This choice does not affect the fact that the leading order approximation is still given by NLS.)

The remainder

$$\frac{\partial R}{\partial t} = \Lambda R + 2\epsilon N(\Psi^{NLS}, R) + \epsilon^{\beta} N(R,R) + \epsilon^{-\beta}\mathrm{Res}(\Psi^{NLS})$$

If we can control the linear evolution then the nonlinear term and the inhomogeneous term can be controlled by Gronwall's inequality.

Thus, our approximation theorem boils down to showing that solutions of the *linear* equation

$$\frac{\partial R}{\partial t} = \Lambda R + 2\epsilon N(\Psi^{NLS}, R)$$

Remain $\mathcal{O}(1)$ for times $0 \leqslant 1 \leqslant \epsilon^{-2}$.

Controlling the linear evolution

$$\frac{\partial R}{\partial t} = \Lambda R + 2\epsilon N(\Psi^{NLS}, R)$$

Note that the evolution due to Λ preserves the H^s norm so that the problems come from the term $2\epsilon N(\Psi^{NLS}, R)$

In principle, this term could cause the linear evolution to grow like

$$e^{C\epsilon t}$$

which over time scales $t \sim \mathcal{O}(\epsilon^{-2})$ would lead to a loss of control of the error R.

We will attempt to remove this term via a normal form transformation - or more accurately we will attempt to transform the equation to

$$\frac{\partial R}{\partial t} = \Lambda R + \mathcal{O}(\epsilon^2) \ ,$$

whose growth can be no worse than $e^{C\epsilon^2 t}$.

The normal form transform

The Fouier transform of the term $2\epsilon N(\Psi^{NLS}, R)$ can be written as

$$\epsilon \hat{N}(\Psi^{NLS}, R)(k) = \epsilon \int \alpha(k, k-m, m) \hat{\Psi}^{NLS}(k-m) \hat{R}(m) dm$$

$$|\alpha(k, k-m, m)| \leqslant C \min(|k|, \sqrt{|k|}) \ .$$

Note that in fact we should sum over the components of R here - we look at this simplified model to try and illustrate the main ideas in this problem with as few technicalities as possible.

To try and eliminate this term we will make a transformation from R to

$$\tilde{R} = R + \epsilon B(\Psi^{NLS}, R)$$

$$\hat{B}(\Psi^{NLS}, R) = \epsilon \int \beta(k, k-m, m) \hat{\Psi}^{NLS}(k-m) \hat{R}(m) dm$$

(See also "Birkhoff normal form for the nonlinear Schrödinger equation" by W. Craig, A. Selvitella, Y. Wang, Rendiconti Lincei-Matematica e Applicazioni (2013).)

The transformation

If we differentiate this equation we find

$$\partial_t \tilde{R} = \partial_t R + \epsilon B(\partial_t \Psi^{NLS}, R) + \epsilon B(\Psi^{NLS}, \partial_t R) \ .$$

From the formula for Ψ^{NLS} we know that $\partial_t \Psi^{NLS} = i\omega(k_0)\Psi^{NLS} + \mathcal{O}(\epsilon)$.

- We can ignore the $\mathcal{O}(\epsilon)$ terms since when we insert them into the expression for $\partial_t \tilde{R}$ we obtain terms $\mathcal{O}(\epsilon^2)$ which we will consistently ignore.
- We should also have a term proportional to $-i\omega(k_0)$ which we ignore for simplicity - it is handled in exactly the same way as the term that is present.

The transformation

Inserting the expression for $\partial_t R$ into this expression we find

$$\partial_t \tilde{R} = \Lambda(\tilde{R} - \epsilon B(\Psi^{NLS}, R)) + \epsilon N(\Psi^{NLS}, R) + \\ + \epsilon B(i\omega(k_0)\Psi^{NLS}, R) + \epsilon B(\Psi^{NLS}, \Lambda R) + \mathcal{O}(\epsilon^2) .$$

Recall that our goal is to eliminate all terms in the equation for $\partial_t \tilde{R}$ up to $\mathcal{O}(\epsilon^2)$, except for $\Lambda \tilde{R}$.

Thus, we search for B such that

$$-\epsilon \Lambda B(\Psi^{NLS}, R)) + \epsilon N(\Psi^{NLS}, R) + \\ + \epsilon B(i\omega(k_0)\Psi^{NLS}, R) + \epsilon B(\Psi^{NLS}, \Lambda R) = 0$$

The resonances

This leads to a formula for the kernel of the transformation of the form:

$$\beta(k, k-l, m) = \frac{\alpha(k, k-l, m)}{-i\omega(k) + i\omega(k_0) + i\omega(m)} .$$

As usual, the actual expression has many more terms since we have to keep track of the behavior of both of the components of R, but all are of this form (just with different combinations of plus and minus sign in the denominator.

The "resonances" are points at which the denominator of this expression vanishes. It's difficult to keep track of both k and m so before we try to bound this expression we simplify somewhat.

The resonances

Recall that

$$\hat{B}(\Psi^{NLS}, R) = \epsilon \int \beta(k, k-m, m)\hat{\Psi}^{NLS}(k-m)\hat{R}(m)dm$$

We can simplify our consideration of the resonance of the normal form transformation by remembering that the Fourier transfrom of $\hat{\Psi}^{NLS}$ is very strongly concentrated around k_0 (and $-k_0$). Since

$$\hat{\Psi}_0(k) = \hat{A}(\frac{k-k_0}{\epsilon}) + \dots$$

Thus, we can approximate $k - m \approx k_0$ or $m \approx k - k_0$. with this approximation the kernel of the normal form transform becomes:

$$\beta(k, k-l, m) = \frac{\alpha(k, k-l, m)}{-i\omega(k) + i\omega(k_0) + i\omega(k-k_0)} .$$

The resonances

$$\beta(k, k-l, m) = \frac{\alpha(k, k-l, m)}{-i\omega(k) + i\omega(k_0) + i\omega(k-k_0)}\ .$$

Note that the denominator of this expression vanishes if:

- $k = 0$
- $k = k_0$

1. The first resonance can be ignored from the fact that $|\alpha(k, k-m, m)| \leqslant C|k|$ for $k \approx 0$. Thus, zero in the numerator *cancels* that in the denominator and β is bounded for k near zero.
2. The resonance at $k = k_0$ is more serious however since the numerator does not vanish there. However, since $k - m \approx k_0$ if $k \approx k_0$, m must be close to zero and we expect that $\hat{R}(m)$ will be very small when $m \approx 0$ because of the fact that the nonlinearity vanishes at wave number zero.

$$\hat{B}(\Psi^{NLS}, R) = \epsilon \int \beta(k, k-m, m)\hat{\Psi}^{NLS}(k-m)\hat{R}(m)dm$$

Loss of smoothness

These ideas, allow us to deal with the resonances in this model. However, there is an additional difficulty that doesn't appear in finite dimensional normal forms problems.

$$\beta(k,k-l,m) = \frac{\alpha(k,k-l,m)}{-i\omega(k)+i\omega(k_0)+i\omega(k-k_0)} .$$

Recall that $\alpha(k,k-l,m) \sim \sqrt{|k|}$ as $|k| \to \infty$, while the denominator,

$$-i\omega(k)+i\omega(k_0)+i\omega(k-k_0) \sim \text{const.}$$

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$$-i\omega(k) + i\omega(k_0) + i\omega(k-k_0) \sim \text{const.}$$

Thus, the linear transformation defined by β "looses half a derivative". i.e

$$B(\Psi^{NLS}, \cdot) : H^s \to H^{s-1/2} .$$

Invertibility of the transformation

We know that

$$\tilde{R} = T(R) = R + \epsilon B(\Psi^{NLS}, R)$$

maps $H^{s+1/2}$ into H^s. We want to show that it is one-to-one on its image and hence invertible.
Suppose one has a transformation which can be written in Fourier variables as

$$\hat{u}(k) = \hat{v}(k) + \epsilon \int \hat{b}(k)\hat{\Psi}(k-m)\hat{v}(m)dm \; ,$$

where

1. $\hat{b}$ is Lipshitz, pure imaginary and $\hat{b}(k) \sim \sqrt{|k|}$,
2. Ψ is smooth and real valued.

Invertibility of the transformation

$$\begin{aligned}\int \overline{\hat{v}(k)}\hat{u}(k) + \hat{v}(k)\overline{\hat{u}(k)} &= 2\int \overline{\hat{v}(k)}\hat{v}(k) + \varepsilon \int \overline{\hat{v}(k)}\hat{b}(k)\hat{\Psi}(k-m)\hat{v}(m)dm\,dk \\ &\quad +\varepsilon \int \hat{v}(k)\overline{\hat{b}(k)}\ \overline{\hat{\Psi}(k-m)}\ \overline{\hat{v}(m)}dm\,dk \\ &= 2\int \overline{\hat{v}(k)}\hat{v}(k) \\ &\qquad +\varepsilon \int \overline{\hat{v}(k)}\hat{\Psi}(k-m)\hat{v}(m)(\hat{b}(k) + \overline{\hat{b}(m)})dk\,dm\end{aligned}$$

Now from the hypotheses on $\hat{b}$ we have

$$|\hat{b}(k) + \overline{\hat{b}(m)}| = |\hat{b}(k) - \hat{b}(m)| \leqslant C|k-m|\ .$$

Invertibility of the transformation

Inserting this estimate into the integral and applying Young's inequality to bound the convolution we find:

$$2\|\hat{v}\|_{L^2}^2 \leqslant 2\|\hat{v}\|_{L^2}\|\hat{u}\|_{L^2} + C\epsilon\|\hat{v}\|_{L^2}\int |\hat{\Psi}(k-m)|k-m|dk$$

If we now use the fact that Ψ is smooth, we have

$$\|u\|_{L^2}^2 \geqslant C\|v\|_{L^2}^2$$

from which we conclude that this transformation is one-to-one (and hence invertible) on it's image.

The error estimates

The method described above allows us to define an invertible transformation, but the loss of smoothness of the transformation creates a new problem:

$$\tilde{R}_t = \Lambda R + \epsilon^2 L(\tilde{R}) + \epsilon^{\beta} \tilde{N}(\tilde{R}) + \epsilon^{-\beta} \mathrm{Res}(\Psi)$$

The problem is that now, $\tilde{N} : H^s \to H^{s-1}$, i.e. it looses a full derivative. This means that standard existence theorems for quasi-linear equations no longer apply.

We use another approach which relies on the fact that we can assume that our initial data is very smooth.

Smoothing the initial data

Because the initial approximation is of the form

$$\Psi_0(x) = \epsilon A(\epsilon x) e^{ik_0 x} + c.c.$$

its Fourier transform is

$$\hat{\Psi}_0(k) = \hat{A}(\frac{k-k_0}{\epsilon}) + \ldots$$

Thus, the Fourier transform of our initial approximation is very strongly localized around $k = \pm k_0$.

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Thus, the Fourier transform of our initial approximation is very strongly localized around $k = \pm k_0$.

We can truncate Fourier transform so that it has compact support:

- without worsening the degree of our approximation,
- and, we obtain an approximating function that is analytic.

The smoothing process

With this in mind, we rewrite

$$\tilde{\hat{R}}(k,t) = \hat{w}(k,t)e^{-|k|(a-b\epsilon^2 t)}$$

We'll then prove that w remains bounded over time scales of $\mathcal{O}(\epsilon^{-2})$.

If we write out the evolution equation for w, we find:

$$\partial_t \hat{w}(k,t) = \Lambda \hat{w} - \epsilon^2 b|k|\hat{w}(k,t) - \epsilon^2 \tilde{L}(\hat{w}) + \epsilon^{\beta}\tilde{\mathcal{N}}(\hat{w}) + \epsilon^{-\beta}\mathrm{Res}(\Psi)$$

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The smoothing term $-\epsilon^2 b|k|\hat{w}(k,t)$ is just sufficient to offset the loss of smoothness coming from the nonlinear term $\epsilon^{\beta}\tilde{\mathcal{N}}(\hat{w})$.

The approximation result

"Theorem": Given any solution $A(X,T)$ of the nonlinear Schrödinger equation, let

$$\Psi^{NLS}(x,t) = \epsilon A(\epsilon(x+c_g t), \epsilon^2 t) e^{i(k_0 x + \omega_0 t)} + c.c. \, .$$

There there exists $C_0 > 0$ and a solution $u(x,t)$ of the original PDE such that

$$\|u(\cdot,t) - \Psi(\cdot,t)\| \leqslant C\epsilon^{3/2}$$

for $0 \leqslant t \leqslant C_0 \epsilon^{-2}$.

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A similar result holds for the actual water wave problem.

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- Is there a more systematic way of classifying the resonances in this problem?
- Is there a way to avoid (or mitigate) the loss of smoothness in the normal form transformation.