Towards Generalized Hydrodynamic Integrability via the Characteristic Variety

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Fields Institute, Toronto

Short talk – Get to the point!

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- Where to find invariant notions of "difficult to integrate?"

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Lorsqu'un système d'équations aux dérivées partielles *en involution* à un nombre quelconque de fonctions inconnues de n variables indépendantes jouit de la propriété que son intégrale générale ne dépend que de fonctions arbitraires d'un argument, il existe r familles de caractéristiques à $n-1$ dimensions, r désignant le nombre des fonctions arbitraires qui entrent dans l'intégrale générale. Chaque intégrale peut être regardée de r manières différentes comme engendrée par des caractéristiques dépendant d'une constante arbitraire. Dans certains cas, le nombre de ces familles de caractéristiques peut être réduit, certaines familles devenant doubles, triples, etc. Ces cas de réduction sont analogues à ceux qui se présentent dans la réduction d'une substitution linéaire à sa forme normale et la recherche des caractéristiques dépend d'ailleurs d'une telle réduction.

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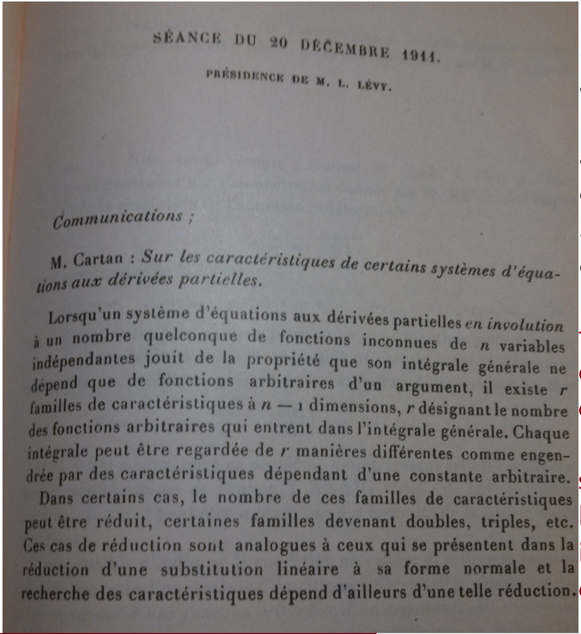

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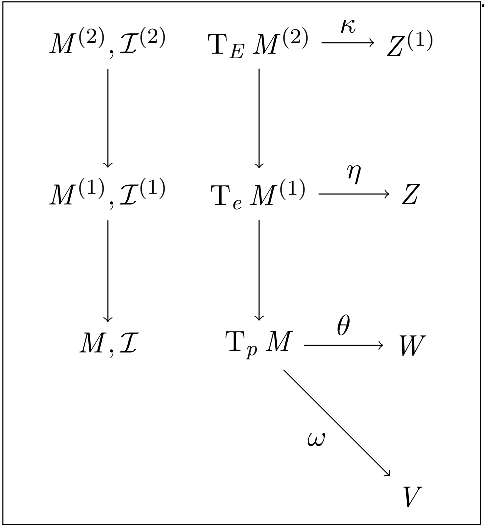
Moreover, those multiplicities in the characteristic variety can be accessed via the incidence correspondence given by the **rank-one variety** of the tableau.

For involutive systems with higher Cartan int, submanifolds secant to the rank-one cone give hydrodynamic reductions, and the secant system indicates **hydrodynamic integrability** in local coordinates.

Two interwoven stories: rank-one variety & hydrodynamic integrability.



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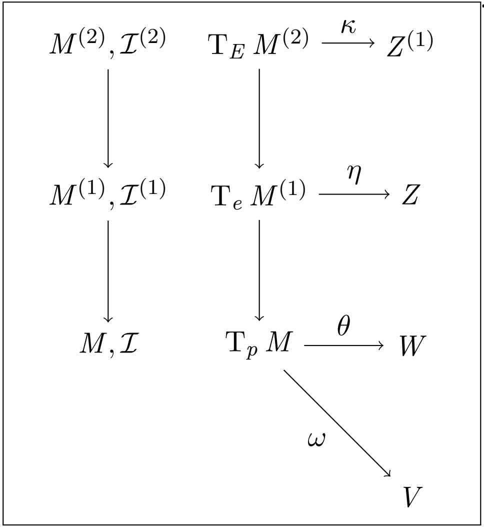

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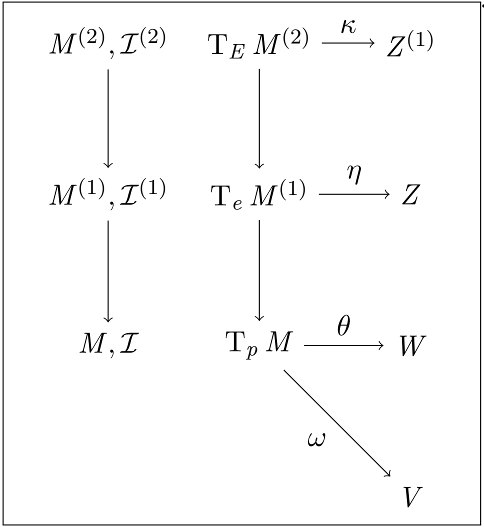

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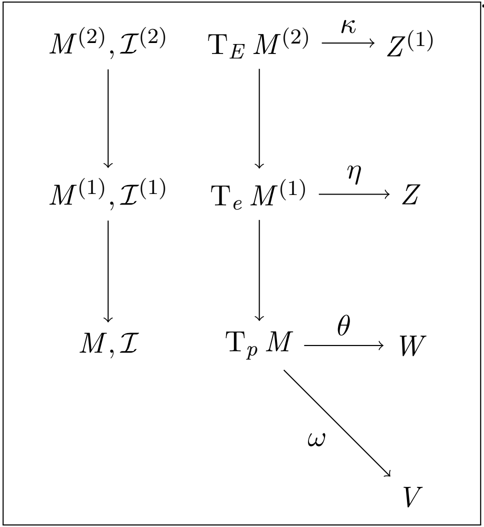

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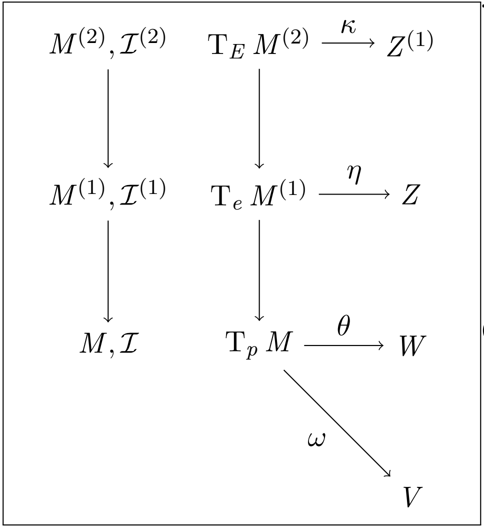

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Characteristic Variety and Rank-One Variety:

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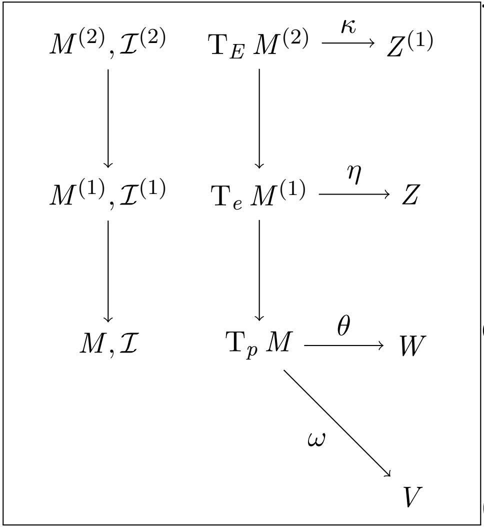

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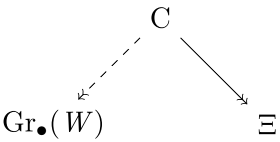
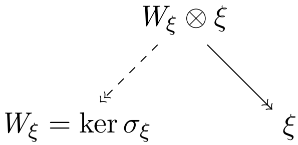
Characteristic Variety and Rank-One Variety:

$$\Xi = \{\xi \in V^* : \exists w, \sigma_\xi(w) = \sigma(w \otimes \xi) = 0\}$$
$$\mathrm{C} = \{z \in Z : \tau(z) = w \otimes \xi, \text{ has rank 1}\}$$

(slides sloppy about $\mathbb{P}$'s)





Some properties if $\mathcal{I}$ is involutive:

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ \mathrm{Gr}_\bullet(W) & & \Xi \end{array} \qquad \begin{array}{ccc} & W_\xi \otimes \xi & \\ \swarrow & & \searrow \\ W_\xi = \ker \sigma_\xi & & \xi \end{array}$$

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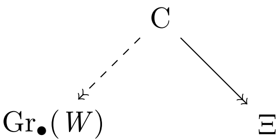

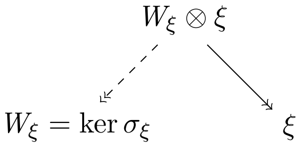

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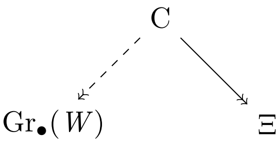

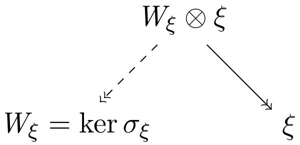

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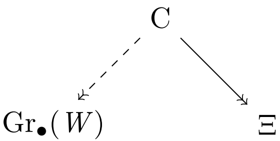

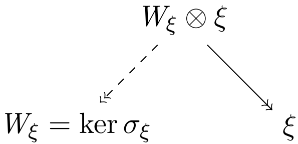

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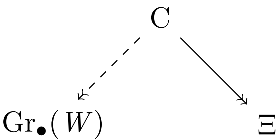

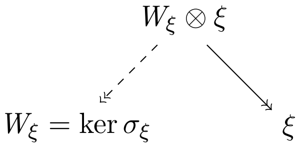

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7. If $\Xi_{\mathbb{R}}$ has appropriate space-like hyperplanes, then $\mathcal{I}$ is hyperbolic.

Some examples with $\dim Z = s = s_1 = \dim Z^{(1)} = 4$

involutive tableau $\iff$ commuting symbol relations $\rightsquigarrow$ compatible primary decompositions.

distinct	duplicates	nilpotents
$\begin{pmatrix} \eta^1 & \lambda_1\eta^1 & \mu_1\eta^1 \\ \eta^2 & \lambda_2\eta^2 & \mu_2\eta^2 \\ \eta^3 & \lambda_3\eta^3 & \mu_3\eta^3 \\ \eta^4 & \lambda_4\eta^4 & \mu_4\eta^4 \end{pmatrix}$	$\begin{pmatrix} \eta^1 & \lambda_1\eta^1 & \mu_1\eta^1 \\ \eta^2 & \lambda_1\eta^2 & \mu_1\eta^2 \\ \eta^3 & \lambda_3\eta^3 & \mu_3\eta^3 \\ \eta^4 & \lambda_4\eta^4 & \mu_4\eta^4 \end{pmatrix}$	$\begin{pmatrix} \eta^1 & \lambda_1\eta^1 + \eta^2 & \mu_1\eta^1 + \eta^2 \\ \eta^2 & \lambda_1\eta^2 & \mu_1\eta^2 \\ \eta^3 & \lambda_3\eta^3 & \mu_3\eta^3 \\ \eta^4 & \lambda_4\eta^4 & \mu_4\eta^4 \end{pmatrix}$
Ξ	Ξ	Ξ
$[1:\lambda_1:\mu_1], [1:\lambda_2:\mu_2], [1:\lambda_3:\mu_3], [1:\lambda_4:\mu_4].$	$[1:\lambda_1:\mu_1], [1:\lambda_1:\mu_1], [1:\lambda_3:\mu_3], [1:\lambda_4:\mu_4].$	$[1:\lambda_1:\mu_1], [1:\lambda_1:\mu_1], [1:\lambda_3:\mu_3], [1:\lambda_4:\mu_4].$
C	C	C
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$\mathrm{Sec}_3(\mathrm{C}) = 4\mathbb{P}^0.$	$\mathrm{Sec}_3(\mathrm{C}) = 1\mathbb{P}^1 \sqcup 2\mathbb{P}^0.$	$\mathrm{Sec}_3(\mathrm{C}) = 1\mathbb{P}^0.$

These systems are easier to distinguish with C than with Ξ.

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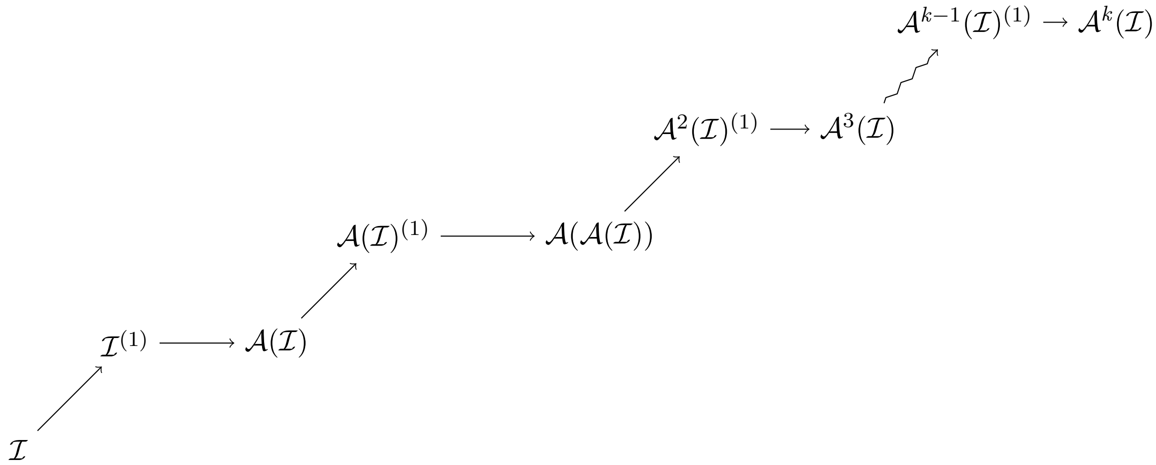

(Where) Does this end?

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That is, integrable systems should be viewed as a subvariety of involutive/regular systems. What is their defining ideal?

Consider this 1st-order system of PDE on functions $(X^n, x^i) \to (Y^r, y^a)$:

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This system is called a *semi-Hamiltonian* or *rich* system of conservation laws (Tsarëv and D.Serre). They:

- are uninteresting in $r \leq 2$.
- describe systems of commuting wavefronts
- admit C^∞ solutions using the generalized hodograph method
- are characterized as orthogonal coordinate webs (Darboux, Tsarëv) (more on this later)
- appear in the linearizations of many “integrable” PDEs (more on this later)

Let $h^a = \frac{\partial y^a}{\partial x^1} \neq 0$, with (h^a) valued in some space H. Consider the EDS on $M = X \times (Y \times H)$ generated by

$$\{\theta^a\} = \left\{ \mathrm{d}y^a - h^a F_i^a(y)\, \mathrm{d}x^i \right\}$$

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3. Best of all possible s_1 involutive systems. Every localization of $\mathcal{A}(\mathcal{I})$ is Frobenius, maximal.

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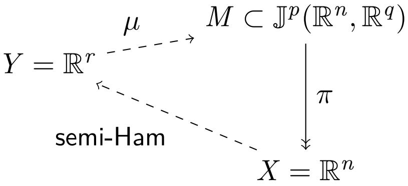

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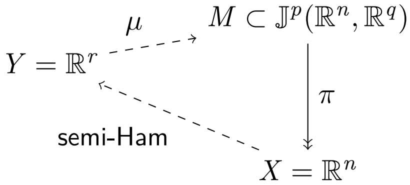

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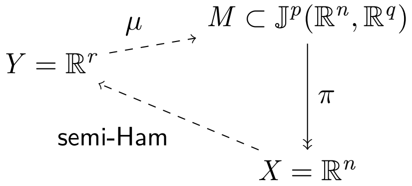

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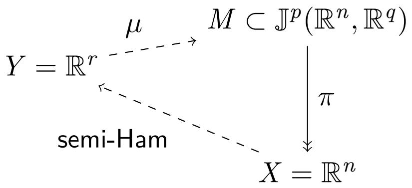

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But can we characterize as EDS with no other restrictions?

Proposition

Suppose that $\mathcal{I}$ is a PDE-type involutive EDS with no Cauchy characteristics or unabsorbable torsion. Then

1. *$\mathcal{I}$ is Frobenius (over $\mathbb{C}$) if and only if $\mathcal{A}(\mathcal{I})$ is always empty. [trivial to prove.]*

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Reminder of the motivation from Lie algebras:

Lie algebras:	trivial	abelian	solvable	semi-simple
	0	$D(\mathfrak{g}) = 0$	$D^k(\mathfrak{g}) = 0$	$D^\infty(\mathfrak{g}) \neq 0$

(But, nothing known about truthfulness of this analogy.)