Recursive Moving Frames

Francis Valiquette

New Paltz
STATE UNIVERSITY OF NEW YORK

Ongoing work with Peter J. Olver

Fields Institute

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Lie pseudo-groups

Lie pseudo-groups $\leftrightsquigarrow$ infinite-dimensional generalization of local Lie group actions

Given $\mathcal{G}$ acting on M, I'm interested in the induced action on $S \subset M$

Example:

$$X = f(x) \qquad Y = e(x,y) = f_x(x)\,y + g(x) \qquad U = u + \frac{e_x}{f_x}$$

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Lie pseudo-groups in action

- symmetry of differential equations
  - Navier–Stokes, Euler, K–P, Davey–Stewartson
- equivalence transformations
  - fiber, point, contact equivalence of differential equations
- gauge transformations
  - Maxwell, Yang–Mills, conformal, string, . . .
- invariant variational calculus – Noether's second theorem (Stay tuned: Irina, Juha)
- . . .

Main theme & tools

Compute

- differential invariants
- invariant differential forms
- invariant differential operators

Tools available:

- Lie's infinitesimal method
- Cartan's method (EDS)
- Equivariant moving frames
- Lie algebroids

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Why use equivariant moving frames?

- Decouples the moving frame theory from reliance on any form of frame bundle or connection
    - basic calculus
    - even an undergraduate student can do this!

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Growing number of applications

- computer vision
- group foliation
- invariant calculus of variation
- invariant geometric flows
- invariant numerical schemes
- computation of
  - Laplace invariants of differential operators
  - invariants and covariants of Killing tensors
  - invariants of Lie algebras

Moving frame algorithm

1. Lie (pseudo-)group action
2. prolonged action (freeness)
3. cross-section
4. normalization
5. invariantization

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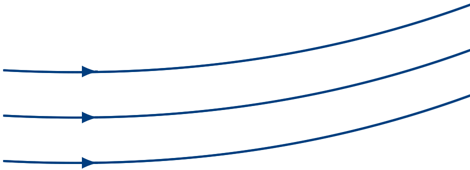

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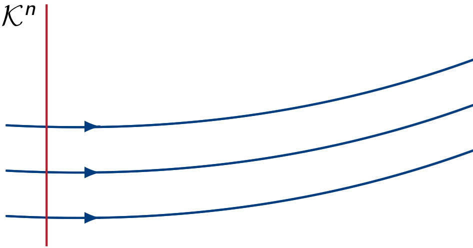

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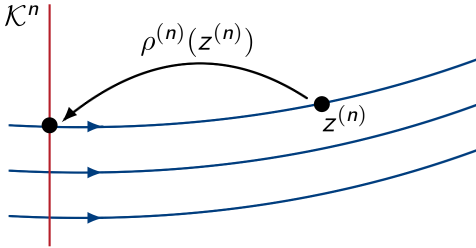

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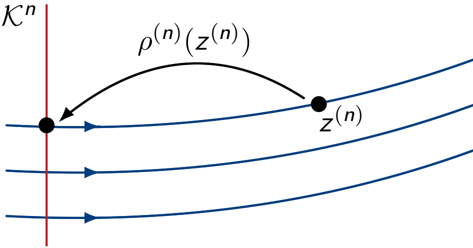

Example

$$X = f(x) \qquad Y = e(x,y) = f_x(x)\,y + g(x) \qquad U = u + \frac{e_x}{f_x}$$

Prolonged action: (Lifted invariants)

$$U_X = \frac{u_x}{f_x} + \frac{e_{xx} - e_x\,u_y}{f_x^2} - 2\frac{f_{xx}\,e_x}{f_x^3} \qquad U_Y = \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} + \frac{f_{xxx} - f_{xx}\,u_y - e_x\,u_{yy}}{f_x^3} - 2\frac{f_{xx}^2}{f_x^4} \qquad U_{YY} = \frac{u_{yy}}{f_x^2}$$

$$U_{XX} = \frac{u_{xx}}{f_x^2} + \frac{e_{xxx} - e_{xx}\,u_y - 2e_x\,u_{xy} - f_{xx}\,u_x}{f_x^3} + \frac{e_x^2\,u_{yy} + 3e_x\,f_{xxx}\,u_y - 4e_{xx}\,f_{xx} - 3e_x\,f_{xxx}}{f_x^4} + 8\frac{e_x\,f_{xx}^2}{f_x^5}$$

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Cross-section:

$$\mathcal{K}^{\infty} = \left\{ x = y = u_{x^k} = u_{yx^k} = 0,\; u_{yy} = 1, \quad k \geqslant 0 \right\}.$$

Normalization equations:

$$X = Y = U_{X^k} = U_{YX^k} = 0 \qquad\qquad U_{YY} = 1$$

Definition: The constant lifted invariants are called phantom invariants

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Solving . . .
$$0 = X = f$$
$$0 = Y = e$$
$$0 = U = u + \frac{e_x}{f_x}$$
$$0 = U_X = \frac{u_x}{f_x} + \frac{e_{xx} - e_x\, u_y}{f_x^2} - 2\frac{f_{xx}\, e_x}{f_x^3}$$
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Definition: The constant lifted invariants are called phantom invariants

$\widehat{\rho}$:

$$\begin{aligned}
f &= 0 \\
e &= 0 \\
e_x &= -u\sqrt{u_{yy}} \\
e_{xx} &= (u\,u_y - u_x)\sqrt{u_{yy}} \\
e_{xxx} &= (u\,u_{xy} + 2u^2 u_{yy} + 2u_x\,u_y - u_{xx} - 2u_y^2)\sqrt{u_{yy}} \\
f_{xx} &= -u_y\sqrt{u_{yy}} \\
f_{xxx} &= (u_y^2 - u_{xy} - u\,u_{yy})\sqrt{u_{yy}} \\
f_x &= \sqrt{u_{yy}}
\end{aligned}$$

Invariantization:

$$U_{XYY} = \frac{f_x\, u_{xyy} - e_x\, u_{yyy} - 2 f_{xx}\, u_{yy}}{f_x^4} \qquad U_{YYY} = \frac{u_{yyy}}{f_x^3}$$

$$\Downarrow$$

$$f_x = \sqrt{u_{yy}} \qquad f_{xx} = -u_y \sqrt{u_{yy}} \qquad e_x = -u \sqrt{u_{yy}}$$

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$$\widehat{U}_{XYY} = \frac{u_{xyy} + u\, u_{yyy} + 2 u_y\, u_{yy}}{u_{yy}^{3/2}} \qquad \widehat{U}_{YYY} = \frac{u_{yyy}}{u_{yy}^{3/2}}$$

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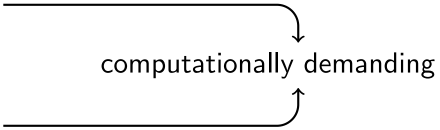

How can steps 2 and 4 be made more effective?

Avoid computing the prolonged action (Like Cartan!)

Recursively construct partial moving frames

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Bundles

$z = (x, u)$ coordinates on M

J^k – submanifold jet bundle:

$$z^{(k)}: \qquad \underset{\text{independent}}{x^1, \ldots, x^p} \qquad \underset{\text{dependent}}{u^1, \ldots, u^q} \qquad \underset{\text{jet}}{\ldots\ u^\alpha_K\ \ldots}$$

$\mathcal{G}$ – Lie pseudo-group:

$$g: \qquad Z^a = \phi^a(z^b) \qquad a, b = 1, \ldots, m = p + q$$

$\mathcal{G}^{(k)}$ – pseudo-group jet bundle:

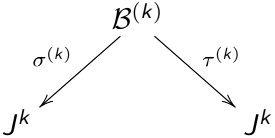
$$g^{(k)}: \qquad \underset{\text{source}}{z^1, \ldots, z^m} \qquad \underset{\text{target}}{Z^1, \ldots, Z^m} \qquad \underset{\text{jet}}{\ldots\ Z^a_B\ \ldots}$$

Lifted bundle

$\mathcal{B}^{(k)}$ – Lifted bundle:

$$z^{(k)} = (\, x^i \;\ldots\; u^\alpha_K \;\ldots\,) \qquad g^{(k)} = (\, X^i_B \;\ldots\; U^\alpha_B \,)$$

Groupoid structure:



$$\sigma^{(k)}(z^{(k)}, g^{(k)}) = z^{(k)}$$
$$\tau^{(k)}(z^{(k)}, g^{(k)}) = g^{(k)} \cdot z^{(k)}$$

Right multiplication:

$$R_h(z^{(k)}, g^{(k)}) = (h^{(k)} \cdot z^{(k)}, g^{(k)} \cdot (h^{(k)})^{-1})$$

Lift

Coframe on $\mathcal{B}^{(\infty)}$:

- Jet forms: $dx^i \qquad du^\alpha_K \qquad \left(\text{or} \quad \theta^\alpha_K = du^\alpha_K - \sum_{k=1}^{p} u^\alpha_{K,k} dx^k \right)$

- Group forms: $\Upsilon^a_B = dZ^a_B - \sum_{b=1}^{m} Z^a_{B,b}\, dz^b$

Jet projection:

$$\pi_J \colon \Omega^*(\mathcal{B}^{(\infty)}) \to \Omega^*_J(\mathcal{B}^{(\infty)}) = \langle \ldots\ dx^i\ \ldots\ du^\alpha_K\ \ldots \rangle$$

Definition: The lift of $\omega \in \Omega^*(J^\infty)$ is

$$\boldsymbol{\lambda}(\omega) = \pi_J[(\tau^{(\infty)})^*\omega]$$

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Invariant coframe

Lifted jet coframe:

$$\sigma^i = \boldsymbol{\lambda}(dx^i) \qquad \sigma^\alpha_K = \boldsymbol{\lambda}(du^\alpha_K) \qquad \left(\text{or } \vartheta^\alpha_K = \boldsymbol{\lambda}(\theta^\alpha_K)\right)$$

Relation:

$$\sigma^\alpha_K = \sum_{i=1}^{p} U^\alpha_{K,i}\sigma^i + \vartheta^\alpha_K \equiv \sum_{i=1}^{p} U^\alpha_{K,i}\sigma^i$$

Lifted jet frame:

$$\mathbb{D}_i = \boldsymbol{\lambda}\left(\frac{\partial}{\partial x^i}\right) \qquad \mathbb{D}^K_\alpha = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u^\alpha_K}\right)$$

defined by $\quad d_J F(x, u^{(n)}, g^{(n)}) = \sum_{i=1}^{p} \mathbb{D}_i(F)\,\sigma^i + \sum_{\alpha,K} \mathbb{D}^K_\alpha(F)\,\sigma^\alpha_K$

Maurer–Cartan forms:

$$\mu^a_B = \mathbb{D}_{b^1}\cdots\mathbb{D}_{b^k}(\Upsilon^a) \qquad B = (b^1,\ldots,b^k) \qquad 1 \leqslant b^i \leqslant m$$

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Partial moving frame

Definition: A partial moving frame of order k is a right-invariant (local) subbundle $\widehat{\mathcal{B}}^{(k)} \subset \mathcal{B}^{(k)}$. Right-invariance

$$R_h(\widehat{\mathcal{B}}^{(k)}) \subset \widehat{\mathcal{B}}^{(k)}$$

Proposition: If

$$T\mathcal{K}^k\big|_{z^{(k)}} \oplus \mathfrak{g}^{(k)}\big|_{z^{(k)}} = TJ^k\big|_{z^{(k)}}$$

for all $z^{(k)} \in \mathcal{K}^k$, then $\widehat{\mathcal{B}}^{(k)} = (\tau^{(k)})^{-1}\mathcal{K}^k$ defines a partial moving frame or order k.

Moving frame:	$\widehat{\rho}^{(k)}(z^{(k)}) \in \widehat{\mathcal{B}}^{(k)}$
Partial moving frame:	$\widehat{\rho}^{(k)}(z^{(k)}, h^{(k)}) \in \widehat{\mathcal{B}}^{(k)}$

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for all $z^{(k)} \in \mathcal{K}^k$, then $\widehat{\mathcal{B}}^{(k)} = (\tau^{(k)})^{-1}\mathcal{K}^k$ defines a partial moving frame or order k.

Moving frame:	$\widehat{\rho}^{(k)}(z^{(k)}) \in \widehat{\mathcal{B}}^{(k)}$
Partial moving frame:	$\widehat{\rho}^{(k)}(z^{(k)}, h^{(k)}) \in \widehat{\mathcal{B}}^{(k)}$

Invariantization

Definition: Let $\widehat{\rho}$ be a (partial) moving frame. The invariantization of $\omega \in \Omega^*(J^\infty)$ is

$$\widehat{\omega} = \iota(\omega) = \widehat{\rho}^*[\boldsymbol{\lambda}(\omega)]$$

Notation:

$$\widehat{\sigma}^i = \iota(dx^i) \qquad \widehat{\sigma}^\alpha_K = \iota(du^\alpha_K) \qquad \widehat{\mathbb{D}}_i = \iota\left(\frac{\partial}{\partial x^i}\right) \qquad \widehat{\mathbb{D}}^K_\alpha = \iota\left(\frac{\partial}{\partial u^\alpha_K}\right)$$

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Recurrence relations

$$\boxed{\iota \circ d \neq d \circ \iota} \qquad \iota = \widehat{\rho}^* \circ \pi_J \circ (\tau^{(\infty)})^*$$

$$\mathfrak{g} = \{\text{local vector fields tangent to pseudo-group orbits in } M\}$$

$$\mathfrak{g}^{(k)} = \{\text{local vector fields tangent to pseudo-group orbits in } J^k\}$$

The prolongation of

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha} \quad \in \quad \mathfrak{g}$$

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$$\mathbf{v}^{(k)} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0}^{k} \varphi_\alpha^K \frac{\partial}{\partial u_K^\alpha} \quad \in \quad \mathfrak{g}^{(k)}$$

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$$\varphi_\alpha^K = D_K\left(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{K,i}^\alpha$$

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New Paltz

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Lift of vector field jets:

$$\boldsymbol{\lambda}(\zeta_B^a) = \mu_B^a$$

Also

$$X^i = \boldsymbol{\lambda}(x^i) \qquad U_K^\alpha = \boldsymbol{\lambda}(u_K^\alpha)$$

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$$dX^i = \sigma^i + \mu^i \qquad dU_K^\alpha = \sigma_K^\alpha + \psi_K^\alpha$$

$$= \sum_{i=1}^{p} U_{K,i}^\alpha \sigma^i + \vartheta_K^\alpha + \psi_K^\alpha$$

where $\psi_K^\alpha = \boldsymbol{\lambda}(\varphi_\alpha^K)$

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Normalized recurrence relations:

$$d\widehat{X}^i = \widehat{\sigma}^i + \widehat{\mu}^i \qquad d\widehat{U}_K^\alpha = \widehat{\sigma}_K^\alpha + \widehat{\psi}_K^\alpha = \sum_{i=1}^{p} \widehat{U}_{K,i}^\alpha \, \widehat{\sigma}^i + \widehat{\vartheta}_K^\alpha + \widehat{\psi}_K^\alpha$$

where $\widehat{\psi}_K^\alpha = \widehat{\rho}^*[\boldsymbol{\lambda}(\varphi_\alpha^K)]$

Recursive moving frame goal

Find

$$\widehat{\rho} \qquad \widehat{\sigma}^i \qquad \widehat{\sigma}^\alpha_K \qquad \widehat{\mathbb{D}}_i \qquad \widehat{\mathbb{D}}^K_\alpha \qquad \widehat{\mu}^a_B$$

recursively without computing

$$\sigma^i \qquad \sigma^\alpha_K \qquad \mathbb{D}_i \qquad \mathbb{D}^K_\alpha \qquad \mu^a_B$$

Rough idea:

normalize, prolong, normalize, prolong, . . .

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Key:

- recurrence relations
- vanishing of (most) group forms along pseudo-group orbits
- $\widehat{\mu}^a_B = \widehat{\mathbb{D}}_{b^1} \cdots \widehat{\mathbb{D}}_{b^k}(\widehat{\mu}^a) \leftrightarrow$ Lie pseudo-group structure equations

By the way (Olver, Pohjanpelto – 2005)

Let $\mathcal{G}$ with determining system

$$F^{(k)}(z, Z^{(k)}) = 0$$

Let

$$L^{(k)}(z, \zeta^{(k)}) = 0$$

be the infinitesimal determining system of $\mathfrak{g}$. Then

$$\lambda[L^{(k)}(z, \zeta^{(k)})] = L^{(k)}(Z, \mu^{(k)}) = 0$$

and the structure equations of $\mathcal{G}$ are

$$d\sigma^a = \sum_{b=1}^{a} \mu_b^a \wedge \sigma^b, \qquad d\mu_C^a = \sum_{b=1}^{m} \left[\sigma^b \wedge \mu_{C,b}^a + \sum_{\substack{C=(A,B) \\ \#B \geqslant 1}} \binom{C}{A} \mu_{A,b}^a \wedge \mu_B^b \right]$$

restricted to $L^{(\infty)}(Z, \mu^{(\infty)}) = 0$

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The Maurer–Cartan structure equations

The Maurer–Cartan structure equations of $\mathcal{G}$ are obtained by restricting the structure equations of $\mathcal{G}$ to a target fiber.

Example 1: The Maurer–Cartan structure equations of

$$X = x + ay + b \qquad\qquad Y = y$$

are

$$d\mu = 0 \qquad\qquad d\mu_Y = 0$$

Example 2: The Maurer–Cartan structure equations of

$$X = f(x) \qquad \text{or} \qquad X = f(x), \quad U = \frac{u}{f'(x)}$$

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$$d\mu_n = \sum_{i=0}^{n} \binom{n}{i} \mu_{i+1} \wedge \mu_{n-i} \qquad n \geqslant 0$$

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Example

Pseudo-group

$$X = f(x) \qquad Y = e(x,y) = f'(x)\,y + g(x) \qquad U = u + \frac{e_x}{f'(x)}$$

Step 1: Order 0 jet forms

$$\sigma^x = f_x\,dx \qquad \sigma^y = e_x\,dx + f_x\,dy \qquad \sigma^u = du + \left(\frac{e_{xx}}{f_x} - \frac{e_x\,f_{xx}}{f_x^2}\right)dx + \frac{f_{xx}}{f_x}dy$$

Order 0 normalizations

$$\mathcal{K}^0 = \{x = y = u = 0\} \qquad \rightsquigarrow \qquad X = Y = U = 0$$

Pseudo-group normalizations

$$f = 0 \qquad e = 0 \qquad e_x = -u\,f_x$$

Group forms

$$\Upsilon_k = df_k - f_{k+1}\,dx \qquad \Psi_k = de_{k,0} - e_{k+1,0}\,dx - f_{k+1}\,dy$$

Normalization

$$\widehat{\Upsilon} = -f_x\,dx \qquad \widehat{\Psi} = f_x(u\,dx - dy) \qquad \widehat{\Psi}_x = -d(u\,f_x) - e_{xx}\,dx - f_{xx}\,dy$$

$\widehat{\Upsilon}_k = 0,\ k \geqslant 1$:

$$f_{k+1} = D_x^k(f_x), \qquad k \geqslant 1$$

$\widehat{\Psi} = 0$:

$$dy = u\,dx \qquad \rightsquigarrow \qquad \frac{dy}{dx} = u(x, y(x))$$

$\underline{\widehat{\Psi}_x = 0:}$

$$e_{xx} = -(u_x + u\,u_y) f_x - 2u\, f_{xx}.$$

$\underline{\widehat{\Psi}_k = 0,\ k \geqslant 2:}$

$$e_{k+1,0} = D_x(e_{k,0})$$

Normalization

$$\widehat{\sigma}^x = f_x\, dx, \qquad \widehat{\sigma}^y = f_x(dy - u\, dx)$$

and

$$\widehat{\sigma}^u = du - \left(u_x + u\, u_y + \frac{u\, f_{xx}}{f_x}\right) dx + \frac{f_{xx}}{f_x} dy \equiv \left(\frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2}\right) \widehat{\sigma}^y = \widehat{U}_Y \widehat{\sigma}^y$$

Order 1 normalization

$$\widehat{U}_Y = 0 \qquad \rightsquigarrow \qquad f_{xx} = -u_y\, f_x$$

Prolongation

$$\begin{aligned}
f &= 0 \\
f_{xx} &= -u_y\, f_x \\
f_{xxx} &= (u_y^2 - u_{xy} - u\, u_{yy}) f_x \\
&\vdots \\
e &= 0 \\
e_x &= -u\, f_x \\
e_{xx} &= (u\, u_y - u_x) f_x \\
e_{xxx} &= (u\, u_{xy} + 2u^2 u_{yy} + 2u_y u_x - u_{xx} - u\, u_y^2) f_x \\
&\vdots
\end{aligned}$$

(Partial) Cross-section

We have normalized

$$f \qquad f_{xx} \qquad f_{xxx} \qquad \ldots \qquad e \qquad e_x \qquad e_{xx} \qquad \ldots$$

without computing the prolonged action!

What is the (partial) cross-section $\mathcal{K}^\infty$ producing these normalizations?

To find $\mathcal{K}^\infty$:

$$\widehat{\rho}\Big|_{\mathcal{K}^\infty} = \mathbb{1}^{(\infty)}\Big|_{\mathcal{K}^\infty}$$

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Substituting

$$f = x \qquad f_x = 1 \qquad f_{xx} = \cdots = 0 \qquad e = y \qquad e_x = \cdots = 0$$

into

$$\begin{array}{llll} f = 0 & f_{xx} = -u_y\, f_x & f_{xxx} = (u_y^2 - u_{xy} - u\, u_{yy}) f_x & \ldots \\ e = 0 & e_x = -u\, f_x & e_{xx} = (u\, u_y - u_x) f_x & \ldots \end{array}$$

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Step 2: Compute the 2nd order partially normalized prolonged action

$$\Downarrow$$

Need the recurrence relations

For

$$X = f(x) \qquad Y = e(x,y) = f_x(x) + g(x) \qquad U = u + \frac{e_x}{f_x}$$

we have

$$\mathbf{v} = \xi(x)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta_x(x,y)\frac{\partial}{\partial u}, \qquad \eta_y = \xi_x$$

Basis of Maurer–Cartan forms

$$\mu_{X^k} = \boldsymbol{\lambda}(\xi_{x^k}) \qquad \nu_{X^k} = \boldsymbol{\lambda}(\eta_{x^k})$$

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\end{aligned}
$$

When

$$\mathcal{K}^1 = \{x = y = u_x = u_y = 0\} \qquad \rightsquigarrow \qquad X = Y = U = U_X = U_Y = 0$$

Recurrence relations:

$$0 = \widehat{\sigma}^x + \widehat{\mu}$$
$$0 = \widehat{\sigma}^y + \widehat{\nu}$$
$$0 = \widehat{\sigma}^u + \widehat{\nu}_X$$
$$0 = \widehat{\sigma}^u_x + \widehat{\nu}_{XX}$$
$$0 = \widehat{\sigma}^u_y + \widehat{\mu}_{XX}$$

When

$$\mathcal{K}^1 = \{x = y = u_x = u_y = 0\} \qquad \rightsquigarrow \qquad X = Y = U = U_X = U_Y = 0$$

Recurrence relations:

$$0 = \widehat{\sigma}^x + \widehat{\mu}$$
$$0 = \widehat{\sigma}^y + \widehat{\nu}$$
$$0 = \widehat{\sigma}^u + \widehat{\nu}_X$$
$$0 = \widehat{\sigma}^u_x + \widehat{\nu}_{XX}$$
$$0 = \widehat{\sigma}^u_y + \widehat{\mu}_{XX}$$

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Normalized Maurer–Cartan forms:

$$\widehat{\mu} = -\widehat{\sigma}^x = -f_x\,dx \qquad\qquad \widehat{\mu}_{XX} = -\widehat{\sigma}^u_y \equiv -\widehat{U}_{YY}\,\widehat{\sigma}^y$$

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We obtain $\widehat{U}_{YY}$ by computing $\widehat{\mu}_{XX} = \widehat{\mathbb{D}}^2_X(\widehat{\mu}) = -\widehat{\mathbb{D}}^2_X(\widehat{\sigma}^x)$

Normalized operator:

$$\widehat{\mathbb{D}}_X = \frac{1}{f_x}\left[\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} + (u_x + u\,u_y)\frac{\partial}{\partial u} + (u_{xy} + u\,u_{yy})\frac{\partial}{\partial u_y} + (u_{xx} + u\,u_{xy})\frac{\partial}{\partial u_x} + \cdots - f_x\,u_y\frac{\partial}{\partial f_x}\right]$$

Fact: $\widehat{\mathbb{D}}_X$ is tangent to the pseudo-group orbits in $\widehat{\mathcal{B}}^{(\infty)}$

$$\frac{dy}{dx} = u, \qquad \frac{du}{dx} = u_x + u\,u_y, \qquad \ldots \qquad \frac{df_x}{dx} = f_{xx} = -u_y\,f_x.$$

Hence

$$\widehat{\mu}_X = \widehat{\mathbb{D}}_X(\widehat{\mu}) = -\widehat{\mathbb{D}}_X(\widehat{\sigma}^x) = u_y\,dx + \frac{1}{f_x}\,df_x$$

$$\widehat{\mu}_{XX} = \widehat{\mathbb{D}}_X(\widehat{\mu}_X) \equiv -\frac{u_{yy}}{f_x^2}\,\widehat{\sigma}^y \qquad (\widehat{U}_{XX} = 0)$$

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$$\widehat{\mu}_X = u_y\, dx + \frac{1}{f_x} df_x, \qquad \widehat{\mu}_{XX} \equiv -\frac{u_{yy}}{f_x^2} \widehat{\sigma}^y$$

Since

$$-\widehat{U}_{YY}\, \widehat{\sigma}^y \equiv \widehat{\mu}_{XX} = -\frac{u_{yy}}{f_x^2} \widehat{\sigma}^y \qquad \Rightarrow \qquad \widehat{U}_{YY} = \frac{u_{yy}}{f_x^2}$$

Case 1 $\Rightarrow u_{yy} = 0$:

$$\widehat{U}_{X^i Y^{j+2}} = 0 \qquad \rightsquigarrow \qquad \text{no further normalization}$$

$$\rightsquigarrow \qquad \text{partial moving frame}$$

Case 2 $\Rightarrow u_{yy} \neq 0$:

$$1 = \widehat{U}_{YY} = \frac{u_{yy}}{f_x^2} \qquad \Rightarrow \qquad f_x = \sqrt{u_{yy}}$$

Then

$$\widehat{\mu}_X = u_y\, dx + \frac{1}{f_x}\, df_x$$

$$\Downarrow$$

$$\widehat{\mu}_X \equiv \left(\frac{u_{xyy} + u\, u_{yyy} + 2\, u_y\, u_{yy}}{2 u_{yy}^{3/2}} \right) \widehat{\sigma}^x + \frac{u_{yyy}}{2 u_{yy}^{3/2}}\, \widehat{\sigma}^y = \frac{\widehat{U}_{XYY}}{2}\, \widehat{\sigma}^x + \frac{\widehat{U}_{YYY}}{2}\, \widehat{\sigma}^y$$

Thus

$$\widehat{U}_{XYY} = \frac{u_{xyy} + u\, u_{yyy} + 2\, u_y\, u_{yy}}{u_{yy}^{3/2}} \qquad \widehat{U}_{YYY} = \frac{u_{yyy}}{u_{yy}^{3/2}}$$

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Thus

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Recursive algorithm

Data is needed:

Lie pseudo-group $\mathcal{G}$ – Recurrence relations – Basis of group forms

Initial step:

- Compute $\sigma^a = d_J Z^a$
- Fix $\mathcal{K}^0$ and solve for pseudo-group parameters
- Prolonged pseudo-group normalizations (vanishing of group forms)
- Compute $\widehat{\sigma}^i,\ \widehat{\sigma}^\alpha \equiv \sum_{k=1}^{p} \widehat{U}^\alpha_k \, \widehat{\sigma}^k$
- Fix $\mathcal{K}^1$ and normalize pseudo-group $\rightsquigarrow$ prolonged pseudo-group normalizations

Loop: $k \geqslant 1$

- Substitute cross-section normalizations into the order k lifted recurrence relations.
- Solve for $\widehat{\sigma}^\alpha_K \quad \leftharpoondown \quad \#K = k$
- Use $\widehat{\mu}^a_{B,b} = \widehat{\mathbb{D}}_b(\widehat{\mu}^a_B)$, and $\widehat{U}^{(k)}$ to obtain coordinate expressions for

$$\widehat{\sigma}^\alpha_K = d\,\widehat{U}^\alpha_K - \widehat{\psi}^\alpha_K \equiv \sum_{i=1}^{p} \widehat{U}^\alpha_{K,k}\,\widehat{\sigma}^k$$

- If possible, normalize some of the $\widehat{U}^\alpha_{K,k} \rightsquigarrow$ pseudo-group normalization $\rightsquigarrow$ prolonged pseudo-group normalization
- corresponding cross-section $\mathcal{K}^{k+1}$
- replace k by $k+1$

Outcomes

- Prolonged action becomes eventually free
  - All pseudo-group parameters are normalized
  - Moving frame
- Prolonged action does not become free
  - Partial moving frame
  - Finitely many unnormalized pseudo-group jets $\leftarrow$ prolonged coframe
  - Infinitely many unnormalized pseudo-group jets $\leftarrow$ involutive coframe

All cases: use recurrence relations to analyze the algebra of differential invariants