Multiroute Flows
&
Node-weighted Network Design

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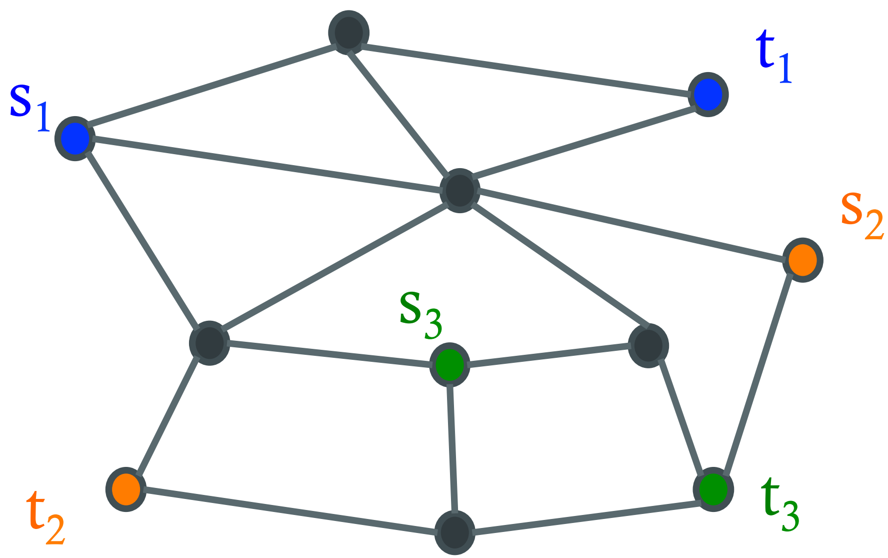
Survivable Network Design Problem (SNDP)

Input:

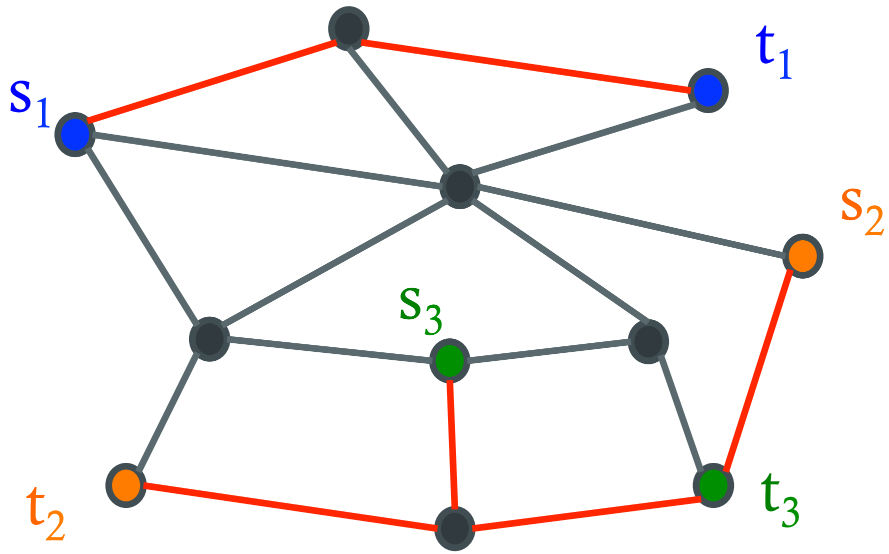
- undirected graph $G=(V,E)$
- integer requirement $r(st)$ for each pair of nodes st

Goal: *min-cost* subgraph H of G s.t H contains $r(st)$ *disjoint* paths for each pair st

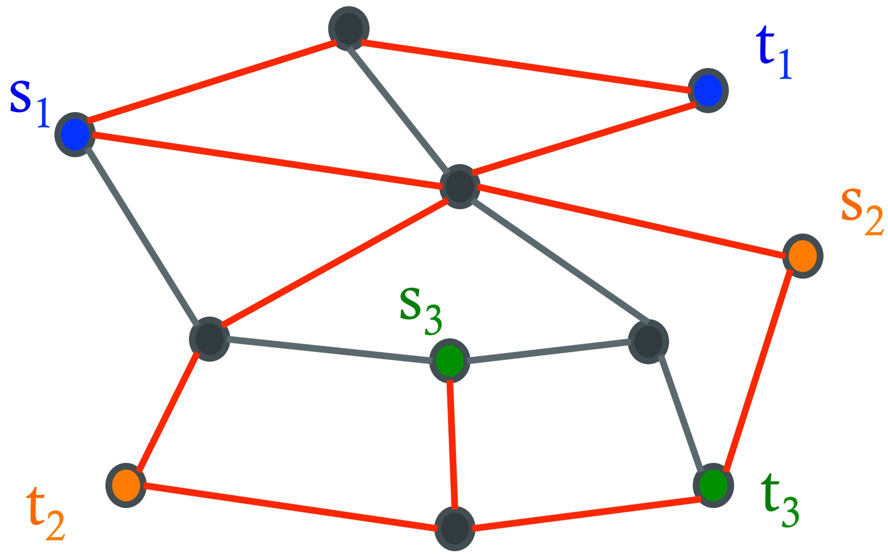


s_1
t_1
s_2
s_3
t_2
t_3

Steiner forest for pairs



$r(s_1t_1) = r(s_2t_2) = 2$ and $r(s_3t_3) = 1$



SNDP Variants

Requirement

- EC-SNDP : paths are required to be edge-disjoint
- Elem-SNDP: element disjoint
- VC-SNDP: vertex/node disjoint

Cost

- edge-weights
- node-weights

Known Approximations

	Edge Weights	Node Weights
Steiner forest	2 - 1/k [AKR'91]	O(log n) [KleinRavi'95]
EC-SNDP	2 [Jain'98]	O(k log n) [Nutov'07]
Elem-SNDP	2 [FJW'01]	O(k log n) [Nutov'09]
VC-SNDP	$O(k^3 \log n)$ [CK'09]	$O(k^4 \log^2 n)$ [CK'09+Nutov'09]

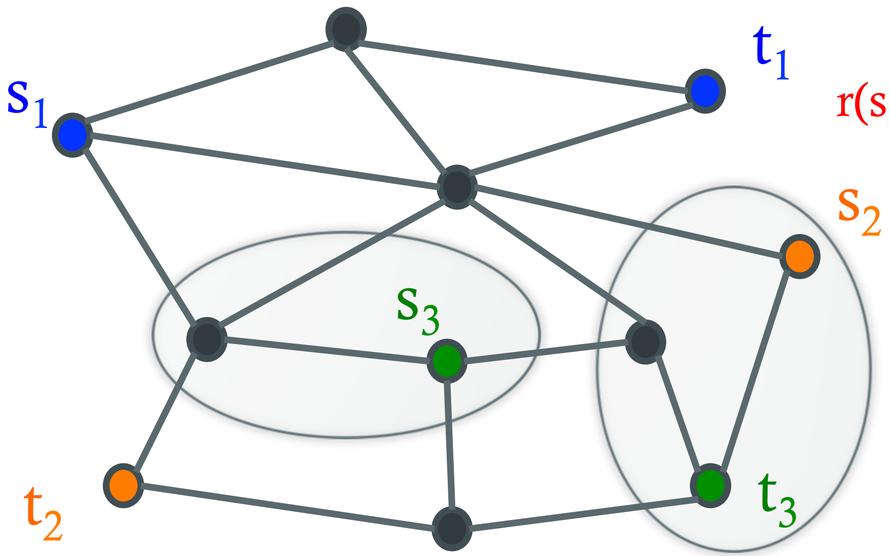
$k := \max_{st} r(st)$

Cut-LP for EC-SNDP

$$\min \sum_e c(e)\, x(e)$$

$$x(\delta(A)) \geq r(st) \qquad A \subset V,\ A \text{ separates } st$$

$$0 \leq x(e) \leq 1$$



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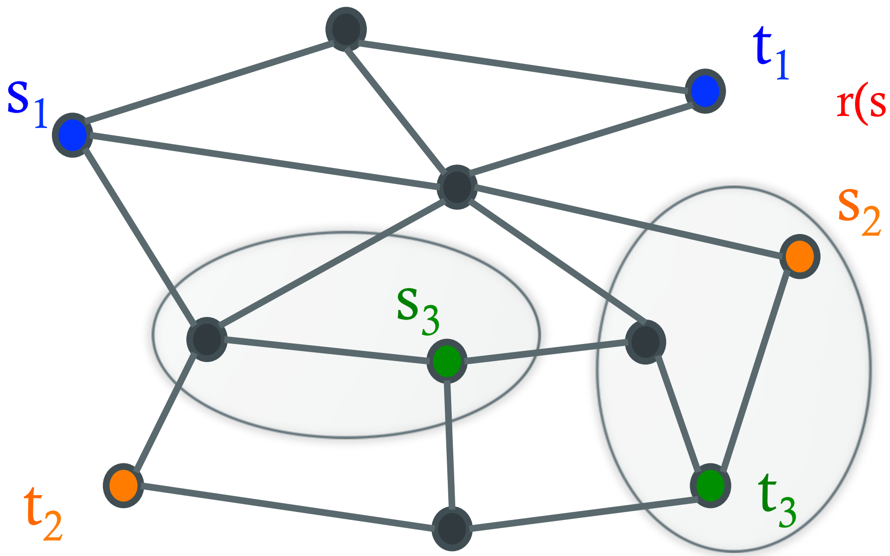
Cut-LP for EC-SNDP



$r(s_1t_1) = r(s_2t_2) = 2$ and $r(s_3t_3) = 1$

Theorem: [Jain] Integrality gap of Cut-LP is 2

Multi-route flows

$\mathcal{P}(st) = \{ p \mid p \text{ is a } st \text{ path} \}$

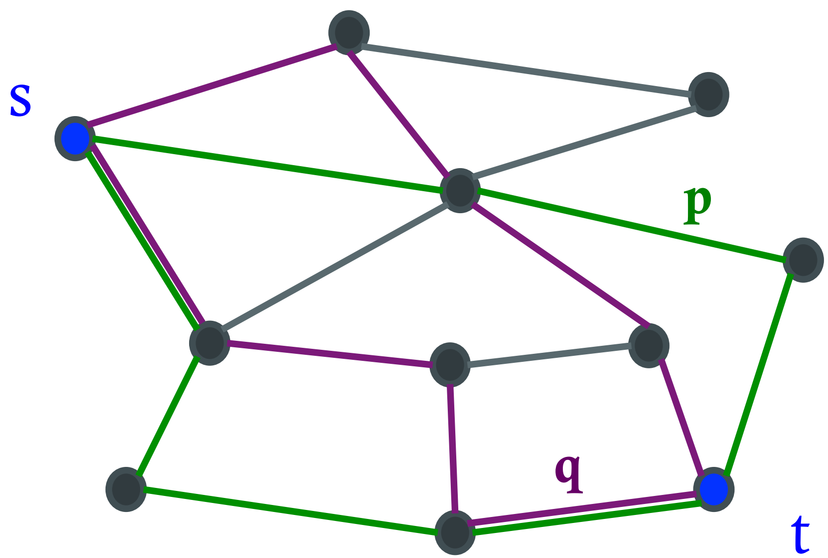
s-t flow, path-based defn $f : \mathcal{P}(st) \rightarrow \mathcal{R}^+$

$f(p)$ flow on path p

$\mathcal{P}(st, h) = \{\mathbf{p} = (p_1, p_2, \ldots, p_h) \mid \text{each } p_j \in \mathcal{P}(st) \text{ and the paths are edge-disjoint} \}$

h-route s-t flow $f : \mathcal{P}(st, h) \rightarrow \mathcal{R}^+$

$f(\mathbf{p})$ flow on path-tuple $\mathbf{p}$

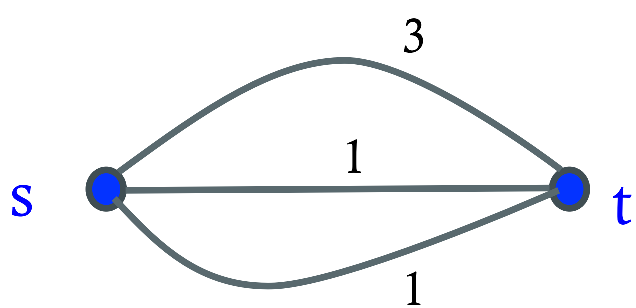

s
p
q
t

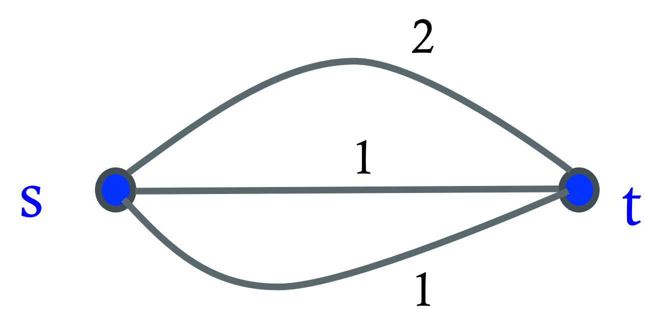
Multiroute flows: basic theorem

[Kishimoto,Aggarwal-Orlin]

Theorem: An acyclic edge s-t flow $x : E \rightarrow \mathcal{R}^+$ with value v can be decomposed into a h-route flow *iff*

$x(e) \leq v/h$ for all edges e





Multi-route flow LP for SNDP

$$\min \sum_e c(e)\, x(e)$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st))} f(\mathbf{p}) \geq 1 \qquad \text{for all } st$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st)):e \in \mathbf{p}} f(\mathbf{p}) \leq x(e) \qquad \text{for all } e, st$$

$$0 \leq x(e)$$

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Solving the LP: Separation oracle for dual is *min-cost* s-t flow

Cut-LP vs Multi-route LP

Claim: Cut-LP and MRF-LP are “equivalent”

Follows from multiroute-flow theorem

Prize-collecting SNDP

Input:

- undirected graph $G=(V,E)$
- integer requirement $r(st)$ for each pair of nodes st
- non-negative penalty $\pi(st)$ for each pair st

Goal: subgraph H of G to minimize $cost(H) + \pi(S)$
where S is set of unsatisfied pairs in H

All-or-nothing: st satisfied if $r(st)$ disjoint paths in H

Prize-collecting SNDP

[BienstockGSW'93] Scaling trick to obtain algorithm for PC-Steiner-tree from Steiner-tree LP

[SSW'07, NSW'08] PC-SNDP for higher connectivity

[HKKN'10] First constant factor for PC-SNDP in all-or-nothing model via "stronger" LP.

Prize-collecting SNDP

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[SSW'07, NSW'08] PC-SNDP for higher connectivity

[HKKN'10] First constant factor for PC-SNDP in all-or-nothing model via "stronger" LP.

Claim: Scaling trick of [BGSW'93] works easily for PC-SNDP via MRF-LP

"stronger" LP of [HKKN'10] equivalent to MRF-LP

MRF-LP for PC-SNDP

$$\mathit{min} \sum_e c(e)\, x(e) + \sum_{st} \pi(st)\, z(st)$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st))} f(\mathbf{p}) \geq 1 - z(st) \qquad \text{for all st}$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st)):e \in \mathbf{p}} f(\mathbf{p}) \leq x(e) \quad \text{for all e, st}$$

$$x(e) \geq 0 \quad \text{for all e}$$

MRF-LP for PC-SNDP

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$x(e) \geq 0$ for all e

Rounding:

- $A = \{\, st \mid z(st) \geq ½ \,\}$
- Pay penalty for pairs in A
- Connect pairs ***not*** in A

MRF-LP for PC-SNDP

$$\min \sum_e c(e)\, x(e) + \sum_{st} \pi(st)\, z(st)$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st))} f(\mathbf{p}) \geq 1 - z(st) \quad \text{for all } st$$

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$$x(e) \geq 0 \quad \text{for all } e$$

Rounding:

- $A = \{ st \mid z(st) \geq ½ \}$
- Pay penalty for pairs in A
- Connect pairs *not* in A

Analysis:

- Penalty for pairs in A is $\leq$ 2OPT
- $x'(e) = \min\{1, 2x(e)\}$ is feasible for MRF-LP to connect pairs **not** in A

MRF-LP for PC-SNDP

Also extends easily to “submodular” penalty functions

Use Lovasz-extension with variables z(st)

([Chudak-Nagano’07] did this for Steiner tree)

Main message: [0,1] variables instead of [0,k] variables

Another "easy" application of multi-route flows

[Srinivasan'99] *Dependent* randomized rounding for multipath-routing to minimize congestion

No need for dependent rounding. [Raghavan-Thompson'87] style independent rounding works with multi-route flow decomposition

Advantages:

- Simpler and transparent
- Allows improvement via Lovasz-Local-Lemma for the short-paths case

Node-Weighted SNDP

Node-Weighted SNDP

[Klein-Ravi'95] Node-weighted Steiner tree/forest

- O(log n) approximation via "spiders"
- Reduction from Set Cover to show Ω(log n) hardness

Node-Weighted SNDP

[Nutov'07,Nutov'09] Node-weighted SNDP

- O(k log n) approximation via generalization of spiders and augmentation framework of [Williamson etal]
- Combinatorial algorithms, not LP based

Advantages of LP-approach

[Guha-Moss-Naor-Schieber'99] LP gap of O(log n) for NW Steiner tree/forest

[Demaine-Hajia-Klein'09] LP gap of O(1) for NW Steiner tree/forest in planar graphs

Via [BGSW'93] similar bounds for NW PC-ST/SF

LP for NW SNDP

Not clear! Why?

LP for NW SNDP

Not clear! Why?

EC-SNDP for a *single pair* is NP-Hard for large k

- $\Omega(\log n)$ hardness: easy reduction from set cover
- [Nutov'07] Related to bipartite k-densest-subgraph problem. Polylog approx unlikely.
- Consequence: Approx ratio depends on k

Open: approximability of single-pair for fixed k

MRF-LP for node weights

$$\mathit{min} \sum_v c(v)\, x(v)$$

$$\sum_{\mathbf{p} \in \mathcal{P}(st,\, r(st))} f(\mathbf{p}) \geq 1 \quad \text{for all } st$$

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Solving MRF-LP for EC-SNDP is hard

MRF-LP can be solved in poly-time for VC-SNDP!

Can solve MRF-LP for EC-SNDP within a factor of k

Integrality gap of MRF-LP

Theorem: Integrality gap of MRF-LP is O(k log n) for EC-SNDP and Elem-SNDP

Theorem: Integrality gap of MRF-LP is O(k) for EC-SNDP and Elem-SNDP on planar graphs

Results extend to VC-SNDP and PC-SNDP via reductions

Approximations for SNDP

	Edge Weights	Node Weights	Node-Weights Planar Graphs
Steiner forest	2 - 1/k [AKR'91]	O(log n) [KleinRavi'95]	O(1) [DHK'09]
EC-SNDP	2 [Jain'98]	O(k log n) [Nutov'07]	O(k)
Elem-SNDP	2 [FJW'01]	O(k log n) [Nutov'09]	O(k)
VC-SNDP	$O(k^3 \log n)$ [CK'09]	$O(k^4 \log^2 n)$ [CK'09,Nutov'09]	$O(k^4 \log n)$

Approx ratios for prize-collecting problems within O(1) for all probs.

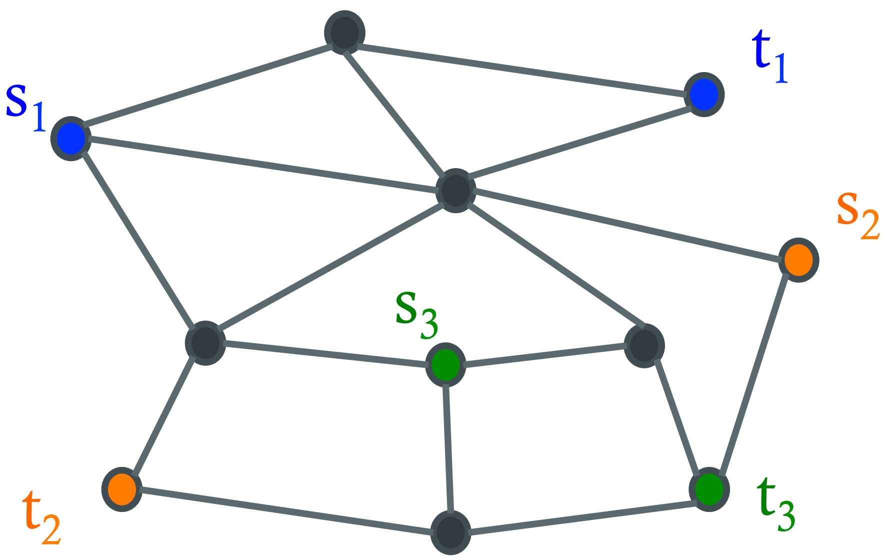
Proving Integrality Gap for MRF-LP

- Augmentation framework [Williamson etal]
- Yet another LP (Aug-LP)
- Spiders and dual-fitting for general graphs following ideas from [Guha etal'99, Nutov'07,'09]
- Primal-dual for planar graphs following [Demaine-Hajia-Klein'09]

Some subtle technical issues

Augmentation Framework

$r(s_1t_1) = r(s_2t_2) = 2$ and $r(s_3t_3) = 1$



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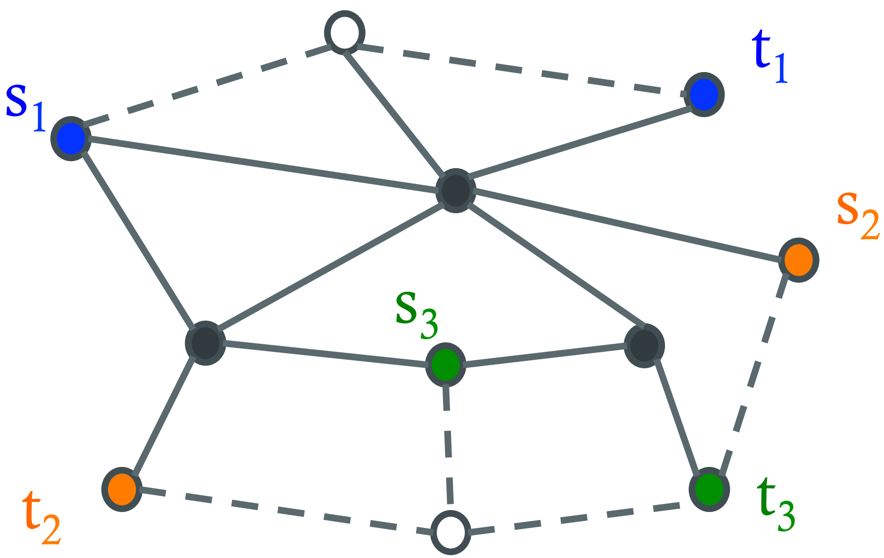

Iteration 1

Node-weighted
Steiner forest problem

Augmentation Framework

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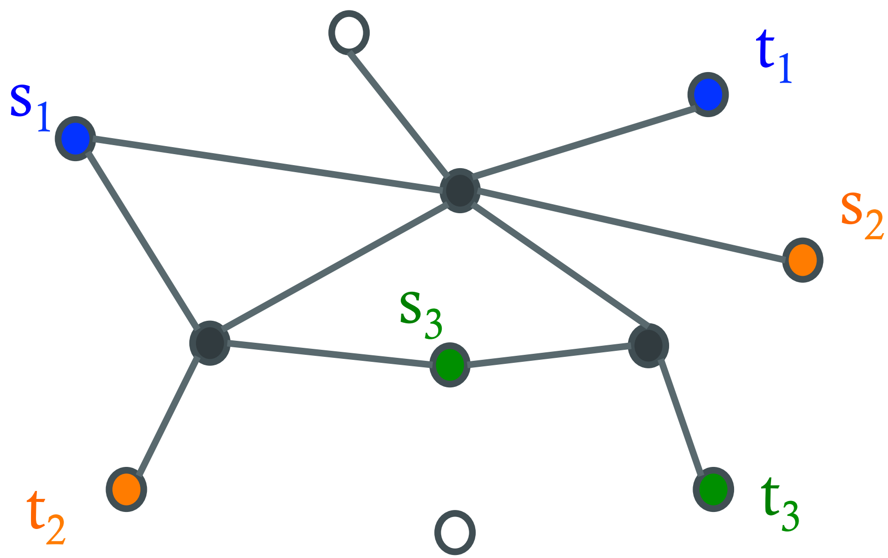

Iteration 2

Increase connectivity by 1 for s_1t_1 and s_2t_2

Residual graph

Covering skew-supermodular function (but arising from proper func) in residual graph

Augmentation Framework

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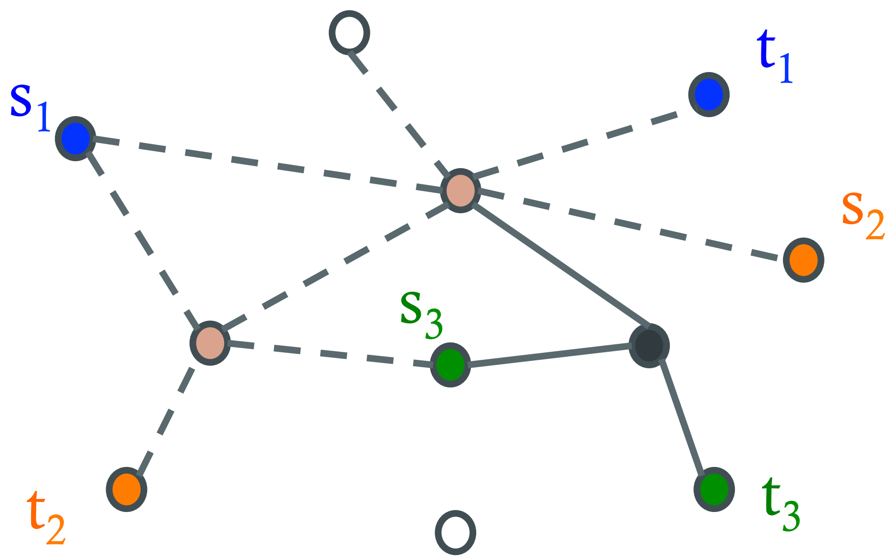

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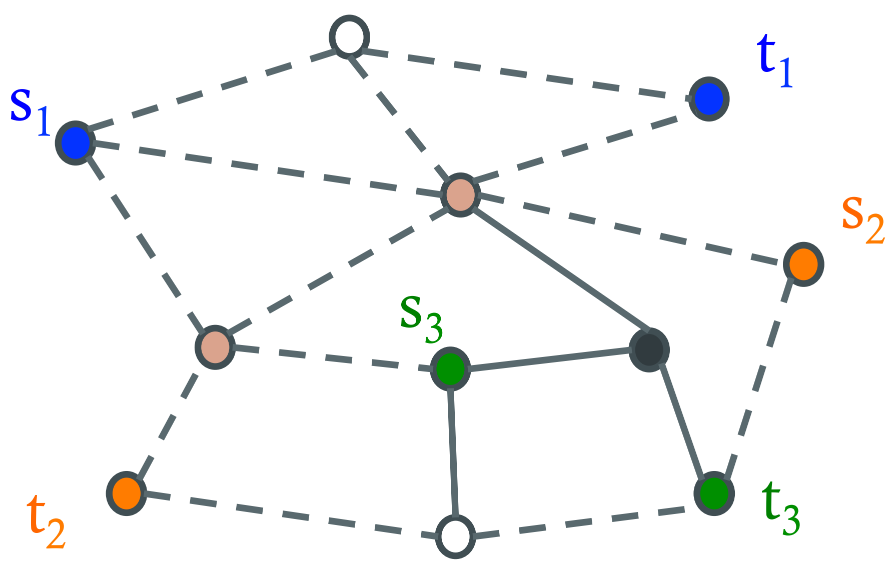


Augmentation Problem

X_{i-1} : nodes selected in iterations 1 to i-1

E_{i-1} : edges in $G[X_{i-1}]$, G_i : residual graph $G \setminus E_{i-1}$

f_i is residual covering function

$f_i(A) = 1$ if A seps st with $r(st) \geq i$ and $|\delta_{E_{i-1}}(A)| = i-1$

Problem: find min-cost set of nodes to cover f_i in G_i

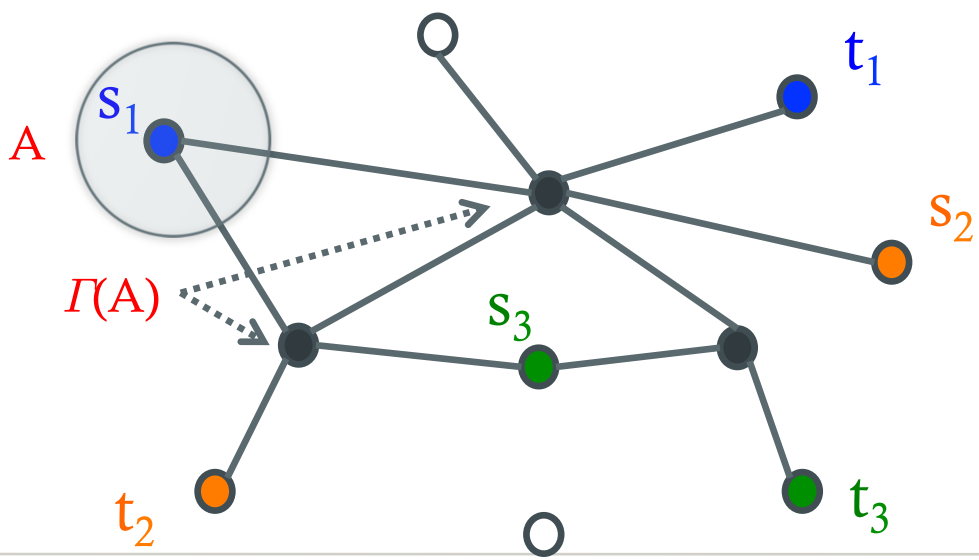
(cost of nodes in X_{i-1} to 0)

Augmentation LP for phase i

$$\min \sum_v c(v)\, x(v)$$

$$\sum_{v \in \Gamma(A)} x(v) \geq f_i(A) \quad \text{for all } A$$

$$x(v) \geq 0 \qquad \text{for all } v$$



Augmentation LP for phase i

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$x(v) \geq 0$ for all v

Theorem: Integrality gap is O(log n) for general graphs and O(1) for planar graphs.

If **(f,x)** is feasible for MRF-LP then **x** is feasible for Aug-LP

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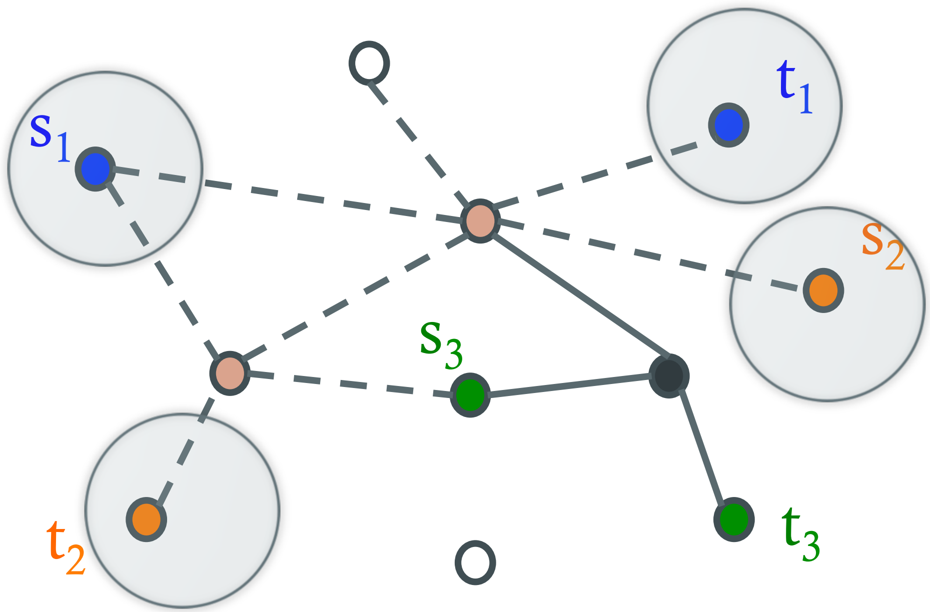
If **(f,x)** is feasible for MRF-LP then **x** is feasible for Aug-LP

Caveat: Integrality gap is unbounded for general skew-supermodular function!

Analysis Aug-LP

- Spiders for general graphs via dual fitting
- Primal-dual for planar graphs
  - Useful lemma on *node-minimal* augmentation

Primal-Dual Analysis

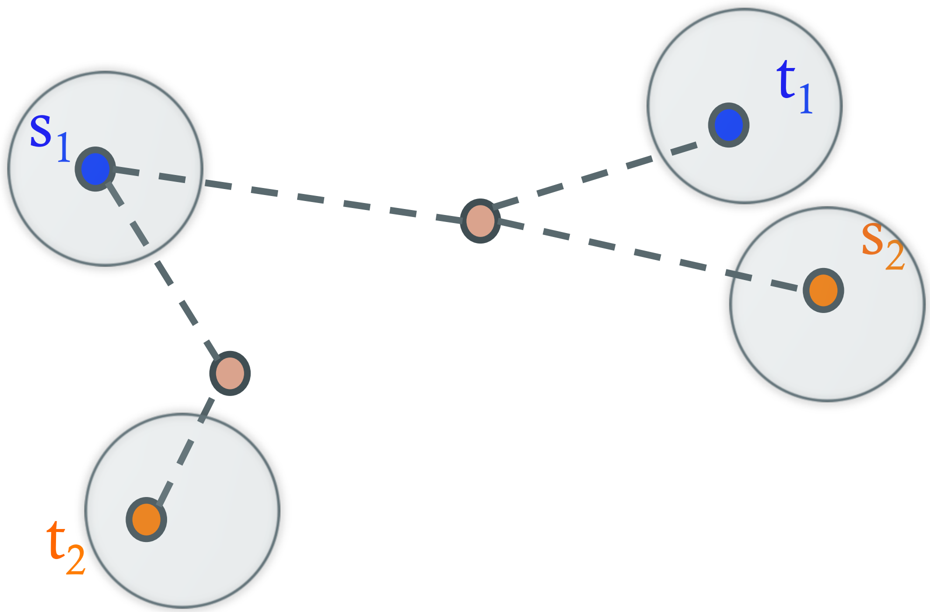


$\mathcal{C}$: minimal violated sets

[Williamson etal] average degree of sets in $\mathcal{C}$ wrt to edges in an *edge-minimal* feasible solution is ≤ 2

Lemma: Number of nodes adjacent to sets in $\mathcal{C}$ in a *node-minimal* feasible solution is at most $4\ |\mathcal{C}|$

Primal-Dual Analysis



$\mathcal{C}$: minimal violated sets

Lemma: Number of nodes adjacent to sets in $\mathcal{C}$ in a *node-minimal* feasible solution is at most $4\ |\mathcal{C}|$

By *planarity* average # of nodes that a set $C \in \mathcal{C}$ is adjacent to is $O(1)$

Thank You!