Chain-constrained spanning trees

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Massachusetts Institute of Technology

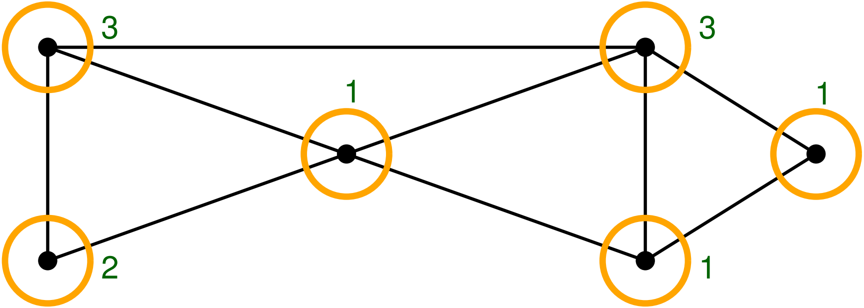
Flexible Network Design 2013
Fields Institute

Degree-bounded spanning trees

Problem

Given: $G = (V, E)$, $c : E \to \mathbb{N}$, degree bounds $b : V \to \mathbb{N}$.

Goal: Find the spanning tree T that satisfies the degree bounds and has minimum cost (if it exists).



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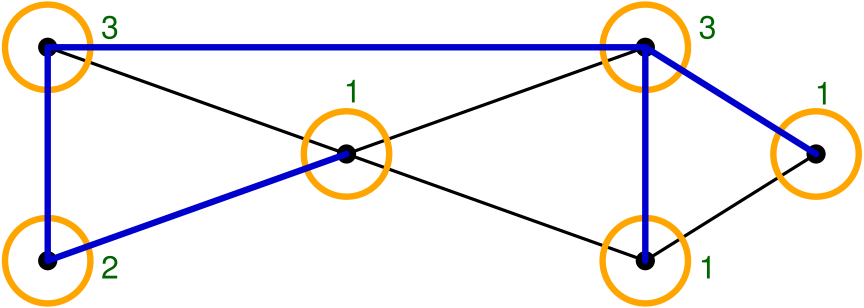
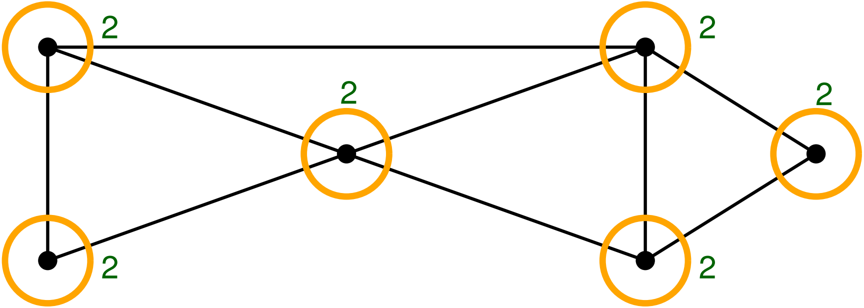


Hardness

NP-hard (even without costs):

if $b(v) = 2$ for all $v \in V$, feasible iff exists a Hamiltonian path.



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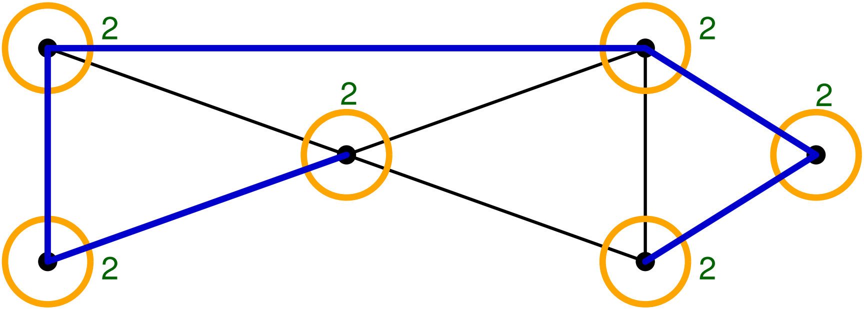


Results for degree-bounded spanning trees

Without costs *Fürer & Raghavachari '94*

If $\exists T^*$ s.t. $\deg_{T^*}(v) \leq b(v)$ for all $v \in V$, produces T with

$$\deg_T(v) \leq b(v) + 1 \quad \forall v \in V.$$

With costs *Goemans '06*

If $\exists T^*$ s.t. $\deg_{T^*}(v) \leq b(v)$ for all $v \in V$, produces T with

$$\deg_T(v) \leq b(v) + 2 \quad \forall v \in V, \quad \textit{and} \quad c(T) \leq c(T^*).$$

An optimal result *Lau & Singh '07*

If $\exists T^*$ s.t. $\deg_{T^*}(v) \leq b(v)$ for all $v \in V$, produces T with

$$\deg_T(v) \leq b(v) + 1 \quad \forall v \in V, \quad \textit{and} \quad c(T) \leq c(T^*).$$

Thin trees

Definition

Given $G = (V, E)$, $z \in P_{\mathrm{ST}}$, a spanning tree T is α-thin if

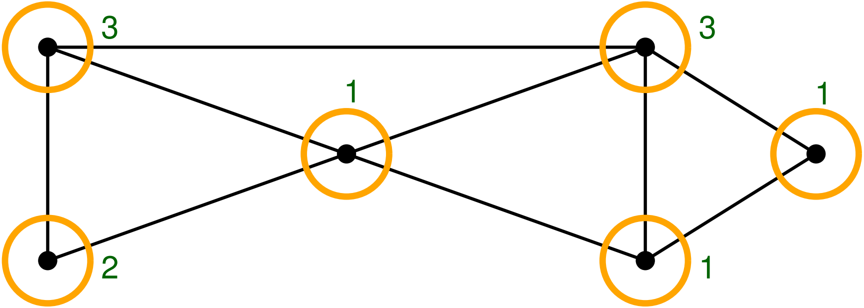
$$|T \cap \delta(S)| \leq \alpha \cdot z(\delta(S)) \quad \forall\, S \subseteq V.$$

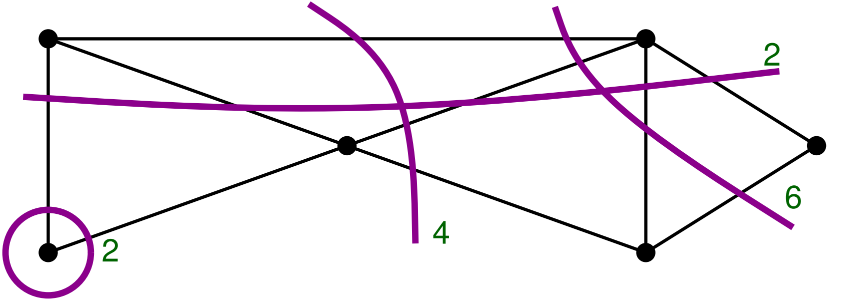
Conjecture *Goddyn*

Every connected graph has a $O(1)$-thin tree.

- Constructive version would imply $O(1)$-approximation for asymmetric TSP

AGMOS '10

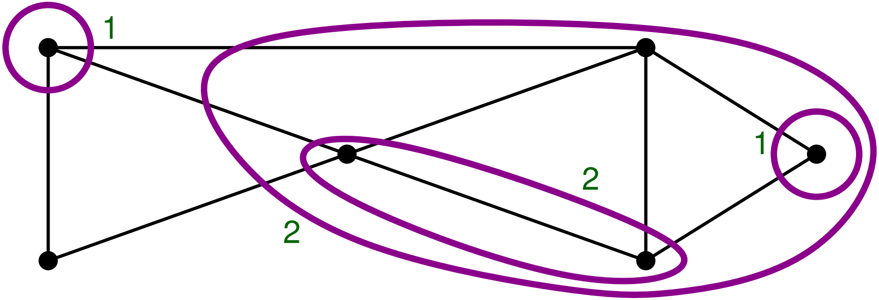

3
3
1
1
2
1


2
4
6
2

Laminar-constrained spanning trees

Given: $G = (V, E)$, $c : E \to \mathbb{N}$, laminar family $\mathcal{F} \subset 2^V$, bounds $b : \mathcal{F} \to \mathbb{N}$.

Goal: Find a spanning tree T satisfying $|T \cap \delta(S)| \leq b(S)$ for all $S \in \mathcal{F}$, of minimum cost.



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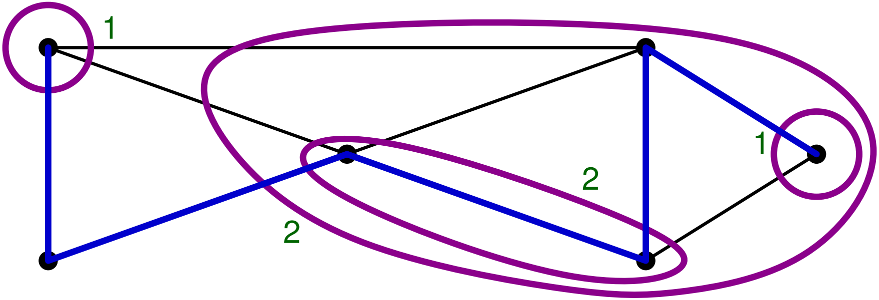

Previous work on the laminar case

Bansal, Khandekar, Könemann, Nagarajan & Peis, IPCO '10

If $\exists T^*$ s.t. $|T^* \cap \delta(S)| \leq b(S)$ for all $S \in \mathcal{F}$, produces T with

$$|T \cap \delta(S)| \leq b(S) + O(\log n) \quad \forall S \in \mathcal{F} \qquad \text{and} \qquad c(T) \leq c(T^*).$$

- Extends iterative relaxation technique.

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Question

Is an additive $O(1)$ violation possible?

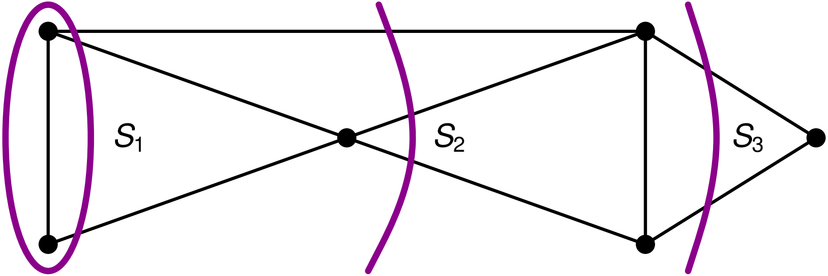
- Bansal et al. prove additive hardness $O(\log^c n)$ for a more general problem, for some $c > 0$.

$$|T \cap E_i| \leq b_i \qquad \text{for an } \textit{arbitrary} \text{ given } E_1, E_2, \ldots, E_t.$$

A negative result

Theorem *O., Zenklusen '13*

Even if $\mathcal{F}$ is restricted to be a chain, obtaining a $o(\frac{\log n}{\log\log n})$-additive violation is NP-hard.



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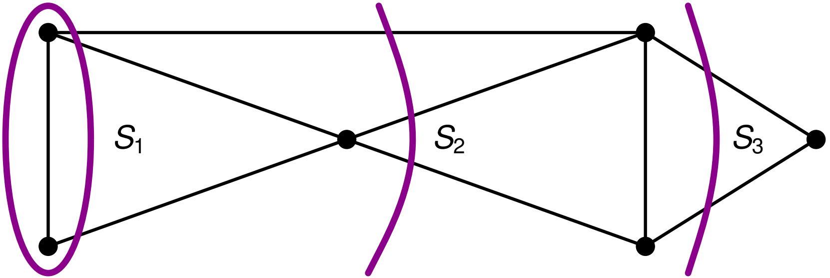
Question

What about an $O(1)$ multiplicative violation?

$$|T \cap \delta(S)| \leq \alpha \cdot b(S) \qquad \forall S \in \mathcal{F}.$$

Chain-constrained spanning trees



Theorem *O., Zenklusen '13*

If $\mathcal{F}$ is a chain, and there exists a spanning tree T^* s.t. $|T^* \cap \delta(S)| \leq b(S)$ for all $S \in \mathcal{F}$, then we can find a spanning tree T (in polynomial time) s.t.

$$|T \cap \delta(S)| \leq 9 \cdot b(S) \qquad \forall S \in \mathcal{F}.$$

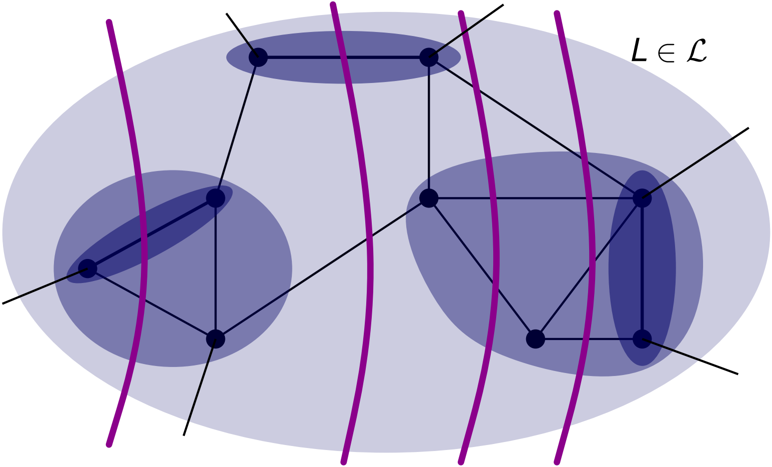
- Based on matroid intersection, not iterative relaxation.

LP relaxation

$$
\begin{aligned}
& \min \sum_{e \in E} c(e)x(e) \\
\text{s.t.} \quad & x \in P_{ST} \\
& x(\delta(S_i)) \leq b(S_i) \qquad \forall\, i \in \{1, 2, \ldots, t\}
\end{aligned}
$$

- Let x^* be an optimum extreme point solution.

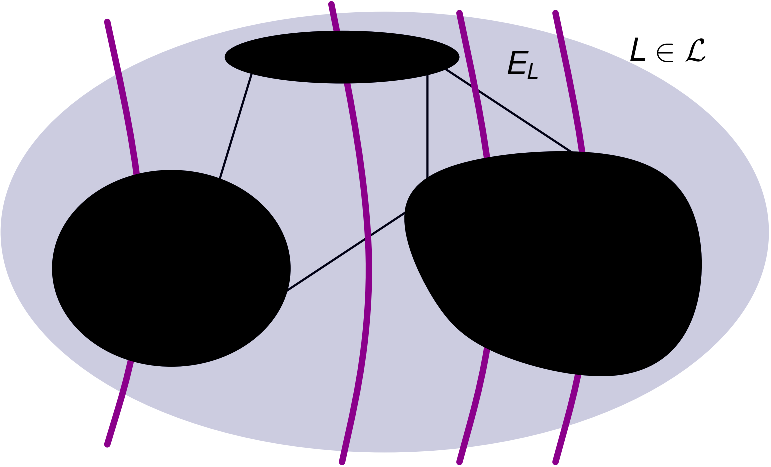
Extreme point structure



Laminar family $\mathcal{L}$ of tight spanning tree constraints:

$$x^*(E(L)) = |L| - 1 \qquad \forall L \in \mathcal{L}.$$

Extreme point structure



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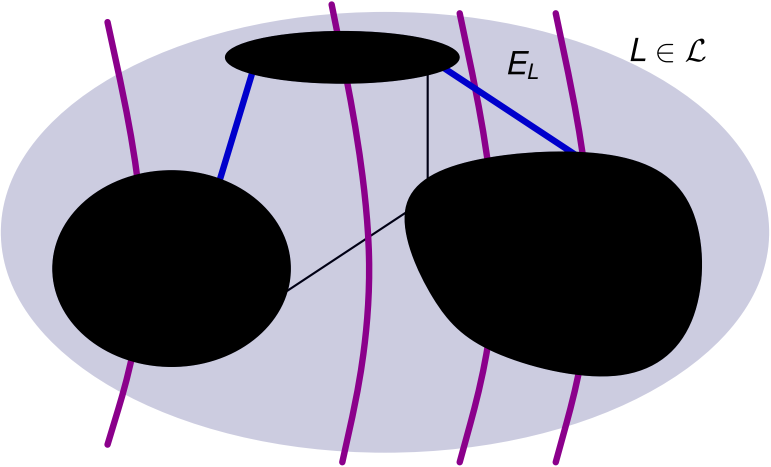

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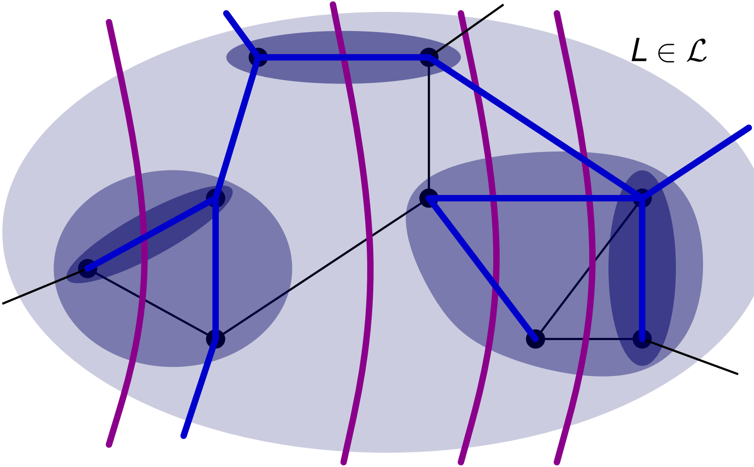

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Subproblems

Naïve attempt

For each $L \in \mathcal{L}$, find spanning tree T_L for L s.t.

$$|T_L \cap \delta(S_i)| \leq C \cdot x^*(\delta(S_i) \cap E_L) \qquad \forall i$$

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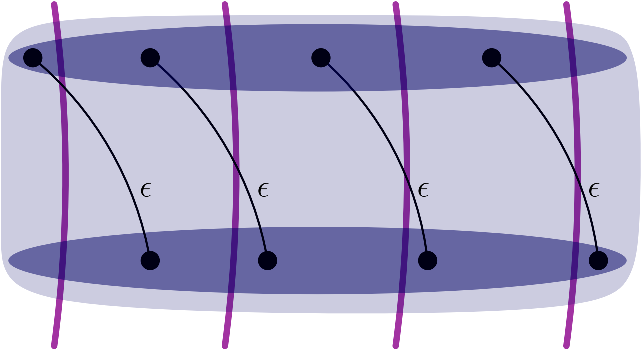
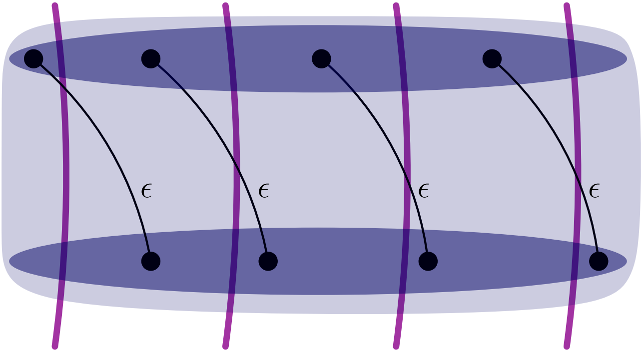


A better plan

Main Lemma

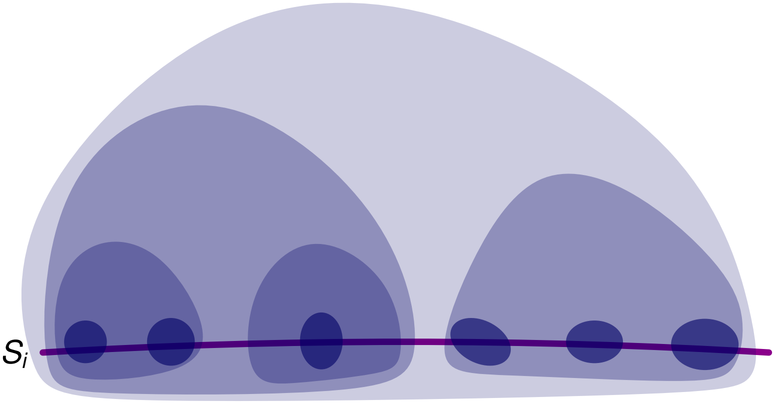
For each $L \in \mathcal{L}$, can find spanning tree T_L for L s.t.

$$|T_L \cap \delta(S_i)| \leq 7 \cdot x^*(\delta(S_i) \cap E_L) + 2 \cdot \mathbf{1}_{\geq 2 \text{ children of } L \text{ cross } S_i} \qquad \forall i.$$



Claim

$$\sum_{L \in \mathcal{L}} \mathbf{1}_{\geq 2 \text{ children of } L \text{ cross } S_i} \leq x^*(\delta(S_i)) \qquad \forall\, i.$$



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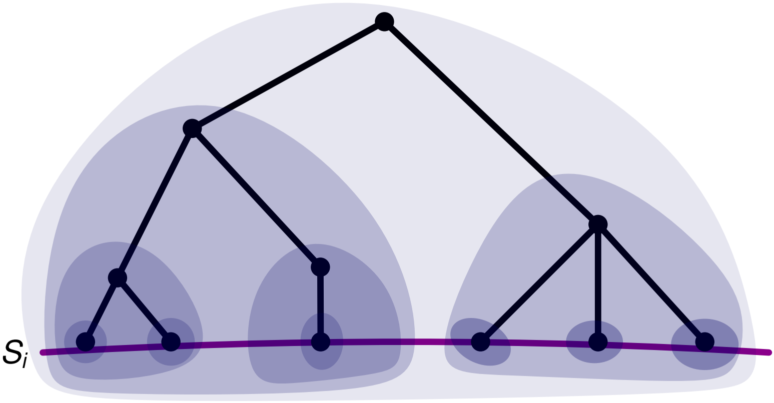

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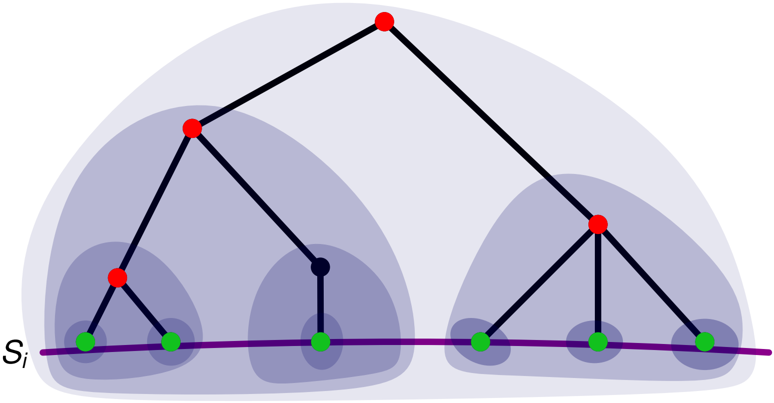

1. $\#$ (red) $\leq$ $\#$ (green)   **2.** $x^*(\delta(S_i) \cap E_L) \geq 1$ for every leaf $L \in \mathcal{L}$

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Main Lemma $\Rightarrow$ Main Theorem:

$$\begin{aligned} |T \cap \delta(S_i)| &= \sum_{L \in \mathcal{L}} |T_L \cap \delta(S_i)| \\ &\leq 7 \sum_{L \in \mathcal{L}} x^*(\delta(S_i) \cap E_L) + 2 \sum_{L \in \mathcal{L}} \mathbf{1}_{\geq \text{2 children of } L \text{ cross } S_i} \\ &\leq 9 x^*(\delta(S_i)). \end{aligned}$$

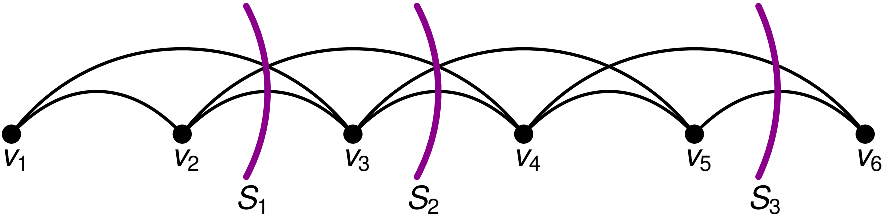
Proof sketch for leaves

Main Lemma for a leaf

If $L \in \mathcal{L}$ is a leaf, can find spanning tree T_L for L s.t.

$$|T_L \cap \delta(S_i)| \leq 7 \cdot x^*(\delta(S_i) \cap E_L) + 2 \cdot \mathbf{1}_{\geq 2 \text{ children of } L \text{ cross } S_i} \quad \forall i.$$

The special objective



$$\min \sum_{\{v_i, v_j\} \in E} |i-j| \cdot x(\{v_i, v_j\})$$

s.t. $x \in P_{ST}$

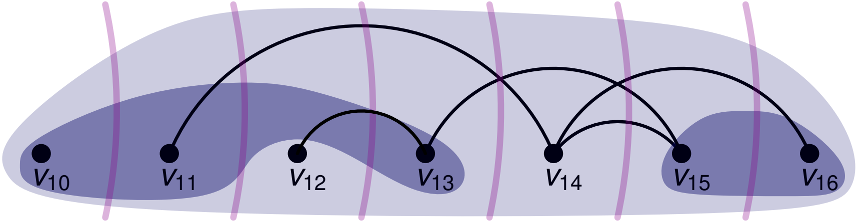
$$x(\delta(S_i)) \leq b(S_i) \qquad \forall\, i \in \{1, 2, \ldots, t\}$$

Rainbow-freeness

$$\min \sum_{\{v_i,v_j\}\in E} |i-j|x(\{v_i,v_j\}).$$

Lemma

For any $L \in \mathcal{L}$, the edge set E_L has no rainbows.



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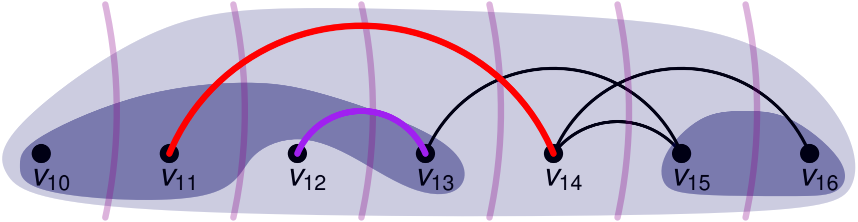

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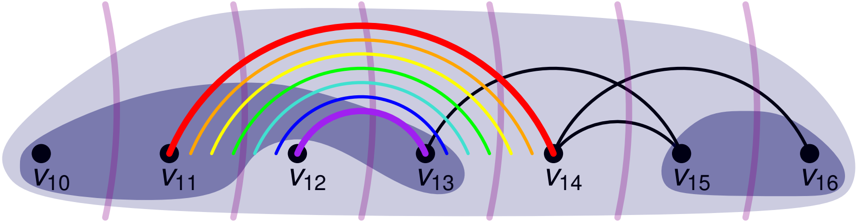

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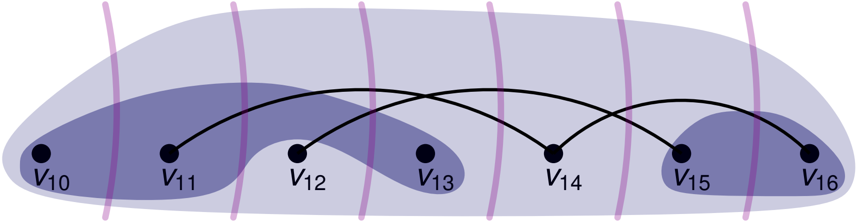
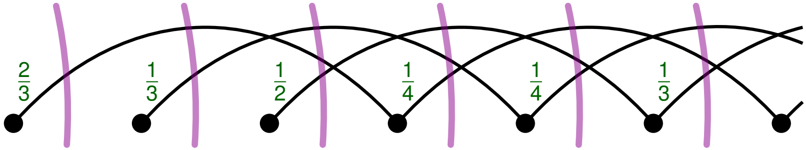


Matroid intersection



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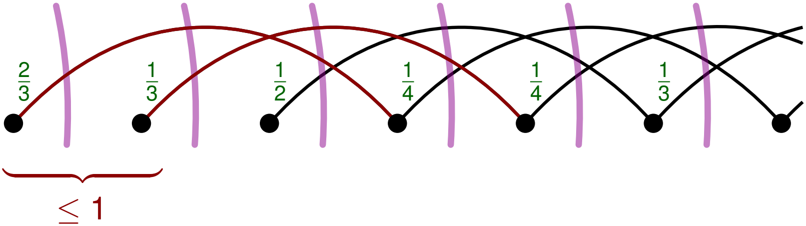

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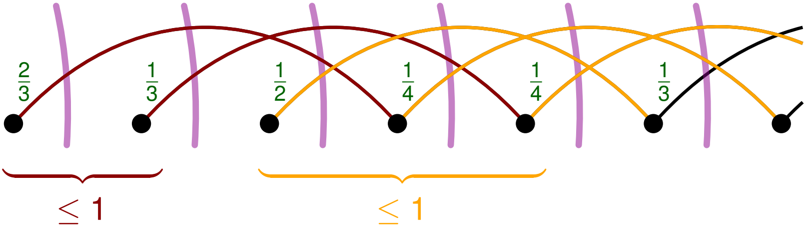

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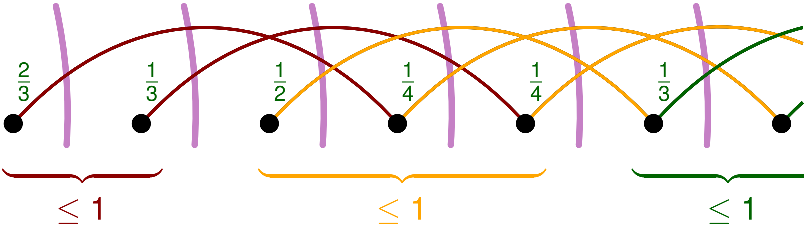

- Define partition matroid $\mathcal{M}_{\text{part}}$ on E_L based on x^*.
- $x^*|_{E_L} \in P(\mathcal{M}_{\text{part}}) \cap P_{\text{ST}(L)}$.

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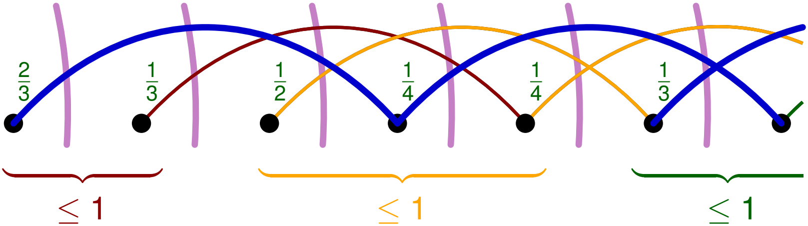

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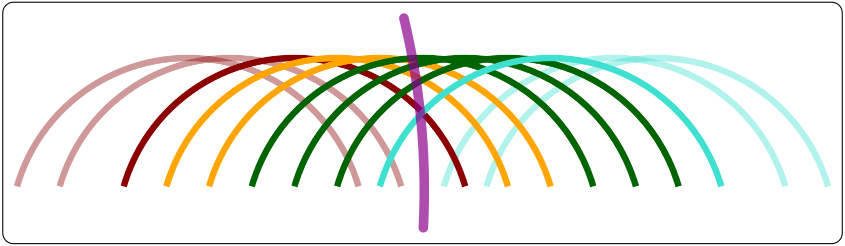

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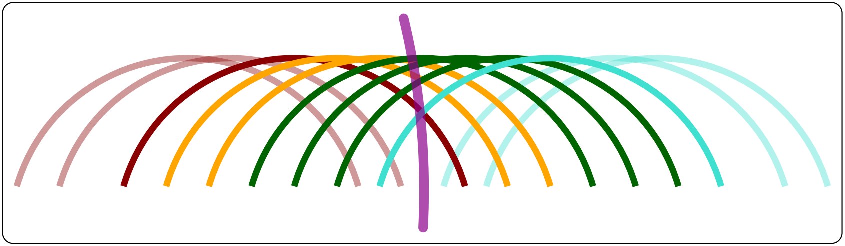

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- Gives T_L s.t. $|T_L \cap \delta(S_i)| \leq x^*(\delta(S_i) \cap E_L) + 2$.
- But $x^*(\delta(S_i) \cap E_L) \geq 1$, so $|T_L \cap \delta(S_i)| \leq 3x^*(\delta(S_i) \cap E_L)$.

Open questions

- Cost version? Even $c(T) = O(c(T^*))$.

- Laminar family of constraints?

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