Higher sumsets with different summands

Let h be a fixed positive integer and $A, B_1, \ldots, B_h$ be finite sets in a commutative group. Suppose that

$$|A + B_i| = \alpha_i |A|.$$

Then the cardinality of the sumset

$$A + B_1 + \ldots + B_h = \{a + b_1 + \ldots + b_h : a \in A, b_i \in B_i \text{ for all } i = 1, \ldots, h\}$$

is known to be bounded by

$$|A + B_1 + \ldots + B_h| \leq \alpha_1 \ldots \alpha_h |A|^{2-1/h}.$$

The upper bound has the correct dependence in $|A|$ and the α_i, yet not in h. It is also of interest to note that there are several ways to prove this inequality, using graph theory, projections and entropy.

We explain how the graph-theoretic approach (developed by Imre Ruzsa) can be modified to yield

$$|A + B_1 + \ldots + B_h| \leq \frac{c_h}{h} \alpha_1 \ldots \alpha_h |A|^{2-1/h},$$

where c_h is an explicit constant that is roughly e^{-1}. This improved upper bound is asymptotically sharp in all the parameters. This is to the best of my knowledge the only instance where the correct dependence in all parameters is known for similar sumset-related problems.

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