EQUILIBRIUM IN COMPETITIVE HELP MODELS IN BIOLOGICAL MARKETS

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OUTLINE

- Current model
- Generalization to N-players
- Computing Nash equilibria (NE) and their analysis
- Stable states (NE)
- Overmatching levels of help
- Agent-based simulations of competitive help
- Generalized Nash game framework and future work
- References

COMPETITIVE HELP MODELS

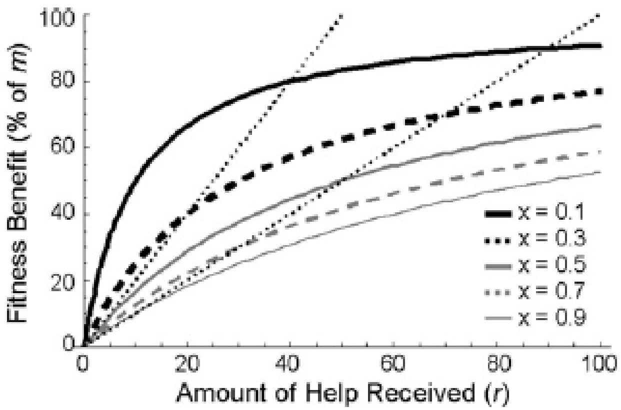
- We look at a population of N biological individuals;
- Proportional matching: They each offer a level of help h_p at a personal cost of h_p and are competing for available help from a mutant individual who offers help at a level h_m.
- this help is available to others according to the attention that the mutant pays to each of them.
- The mutant competes with N-2 individuals for the help provided by another member of the population over the attention of each of the N-1 other members.
- This implies that the total amount of help received is

$$r = \frac{(N-1)h_p(h_m)}{h_m+(N-2)h_p} \qquad (1)$$

- The fitness function W_m for a mutant is "revenue – cost", where

$$revenue := \frac{mr_m}{mx + r_m} - h_m$$

Where $\frac{mr_m}{mx+r_m}$ is diminishing marginal return function: the benefit extracted from larger and larger amounts of help decreases; x is a parameter that shapes the function



- Figure extracted from: Barclay, P. 2011, *Competitive helping increases with the size of biological markets and invades defection* ,Journal of Theoretical Biology 281 (2011) 47–55
- Idea: organisms needing small amounts of help would have smaller values of x.
- Example: helping others by providing opportunities for social learning - x is low if one can learn something from someone else by observing them do it just once;

- The Matching Law in psychology is formulated as:

animals respond to different sources of food or other reinforcement (A and B) at rates which are approximately proportional to the relative rates of reinforcement (i.e., frequency of food availability) => responding in this fashion maximizes their intake.

- Strictest form: $\frac{R_A}{R_A+R_B} = \frac{r_A}{r_A+r_B}$, where R_A, R_B are rates of responding to $A \text{ } and \text{ } B$ and r_A, r_B are the rates of reinforcement
- In the model: $A = h_m, B = h_p$;
- Let p be the proportion of time an individual from the population spends attending to the mutant, which leaves $(N-2)(1-p)$ to attend to all others in population
- Using derivation existing in literature (Baum 1981), it is shown that the optimal investment $p^* = \frac{A}{A+(N-2)B}$ which produces the proportional matching from formula (1) above.

- Overmatching: it is shown that the optimal investment

$$p^* = \frac{A^z}{A^z + (N-2)B^z}$$

produces an overmatching of partnering : z is the degree of matching(i.e., degree to which one receives help in proportion to the relative amount of help one provides);

- Williams 1988: $z = 0$: *individuals cannot choose who to associate with*

  all receive equal attention and help from others regardless of how much they produce

  $z = 1$: *individuals can freely and perfectly choose partners based on how much help each produces; all receive help in proportion to relative amounts they give*

  $z = \infty$: *winner takes all system: the most generous receives all help from others*

KNOWN RESULTS

- Help invades defection (defecting = $h = 0$)
- It is an ESS (there is at least an $h^* > 0$ for some individuals)
- $h^* > 0$ depends on the size N;
- All individuals that are non-mutant are the same, and produce the same level of help.
- The above hold true for overmatching: $z > 1$.

- The above are extrapolated from optimizing the fitness of a mutant; there is no proof/modelling for the ESS claim or overmatching cases.

N-PLAYER SETUP

- We generalize the model to N players with

$$r_i = \sum_{j\neq i, j=1}^{N} h_j\left(\frac{h_i^z}{total_{help} - h_j^z}\right), i = 1, \dots, N,$$

$$total_{help} := \sum_{i=1}^{N} h_i^z$$

Whose fitness functions are

$$W_i(h_1, \dots, h_N) := \frac{mr_i}{mx + r_i} - h_i, \qquad i = 1, \dots, N$$

- Now each player seeks to

$$\max W_i$$
$$s.t. h_i \in [0, m]$$

- A NE point is a vector $h^* \in [0, m]^N$ $so\ that\ for\ each\ i\ we\ have$:

$$W_i\left(\widehat{h_i^*}, h_i\right) \geq W_i(h^*), \qquad \forall h_i \in [0, m], \qquad where$$
$$\widehat{h_i} = (h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_N).$$

SOLVING THE N-PLAYER GAME

- One way is to check whether the payoffs (W_i)

Have good properties that could ensure a more theoretical treatment of the game;

- We would need them to be class C^1 and concave w.r.t. each variable so that the game can be equivalently transformed into a variational inequality problem – NO
- We can look to solve the game with the reaction curves method, i.e., solve system

$$\begin{cases} \dfrac{\partial W_1}{\partial h_1} = 0 \\ \quad \cdots \\ \dfrac{\partial W_N}{\partial h_N} = 0 \end{cases} \quad \text{subject to h} \in [0, m]^N$$

- We can find the solutions of this nonlinear system by recognizing they are equilibrium points of a projected system

$$\frac{dh}{dt} = P_{T_K(h(t))}\left(F\left(h(t)\right)\right), h(0) \in \mathrm{K}, K = [0, m]^N$$

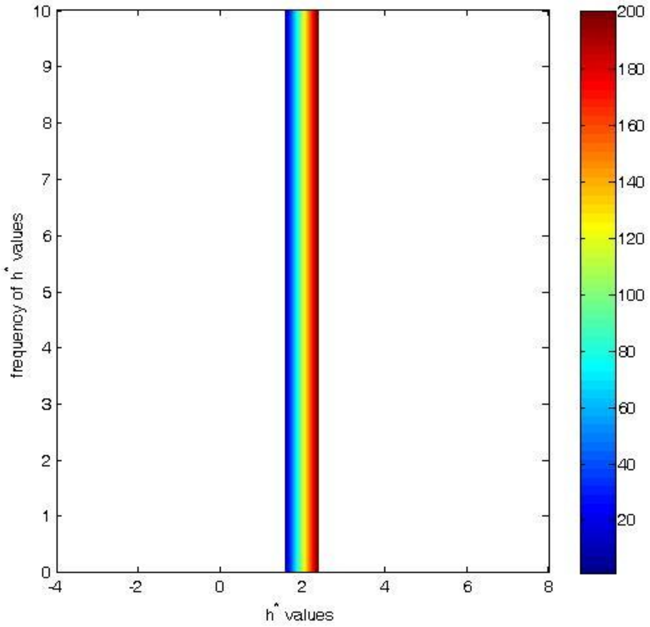
And

$$F(h) \coloneqq (\dots, F_i(h) \coloneqq \frac{\partial W_i}{\partial h_i}(h), \dots)$$

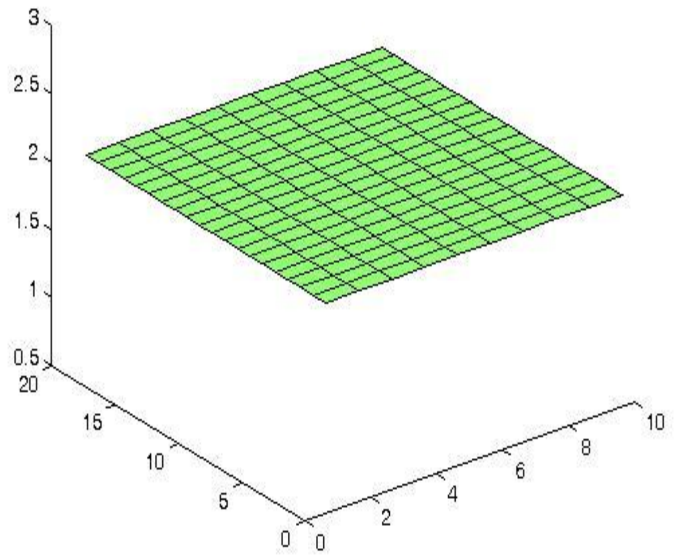
- We have that solutions for this system exist and are unique from each initial point.
- The conditions that we would need to see if its equil points are unique are not satisfied!
- SO: we need to check things numerically, by sweeping the set of initial conditions and trying to detect the set of NE points that emerges.
- Drawback: there may be Nash points which are unstable equil points of the system above; they are unlikely to be found this way.

- Since we are interested to verify the claims in the original paper in this more generalized setting, we look at the emergence of NE solutions and their structure, as we vary N the population size, and z the matching coefficient: we keep m=10, x=0.4 throughout.

N=10 players, z=1, 10 runs over
UD initial points in [0,10]^10



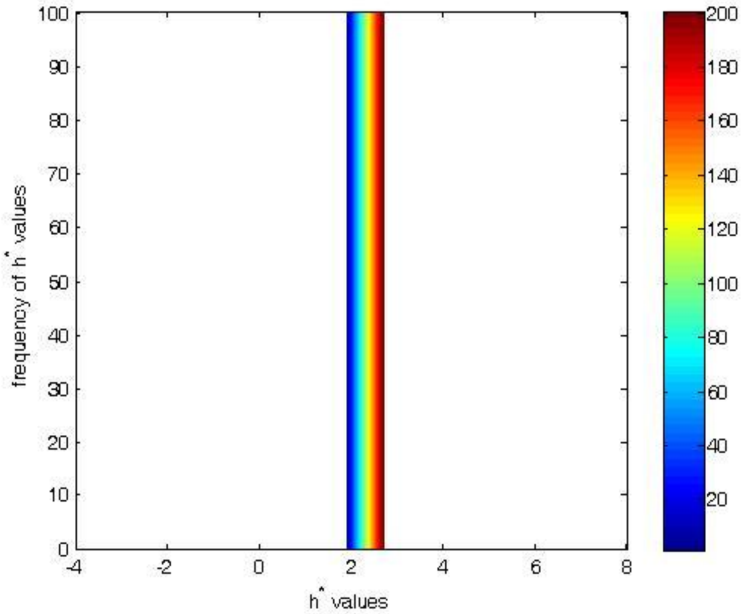
all values are 1.963

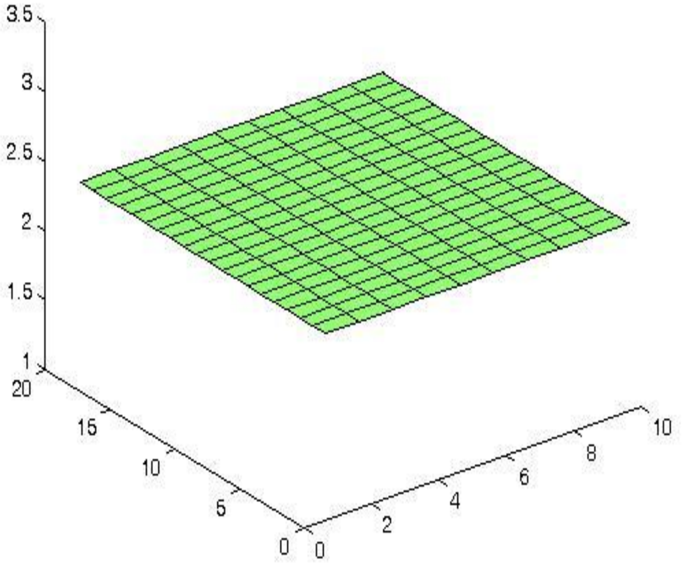


h^* average values over initial pts sweep

- N=50 players, z=1, 10 runs over

UD initial points in [0,10]^50



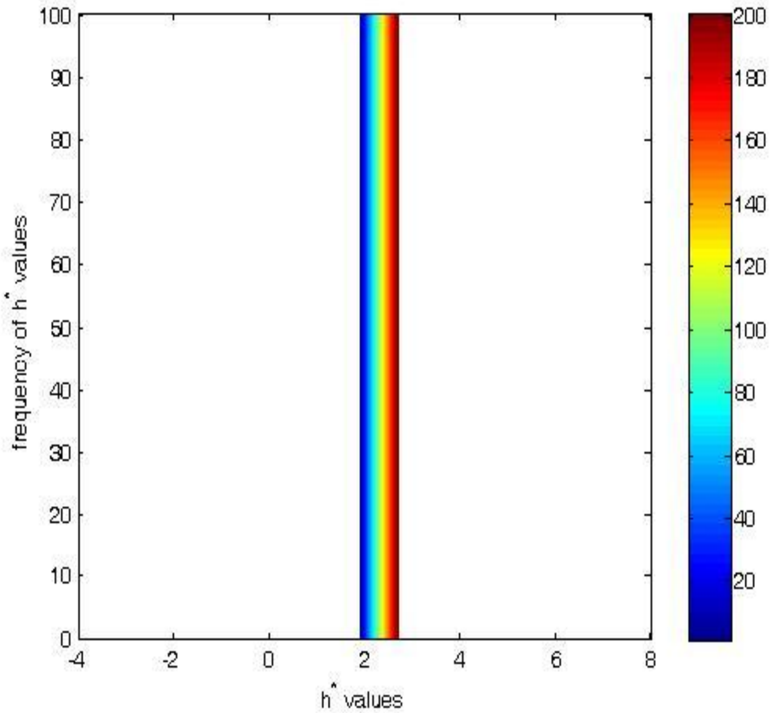


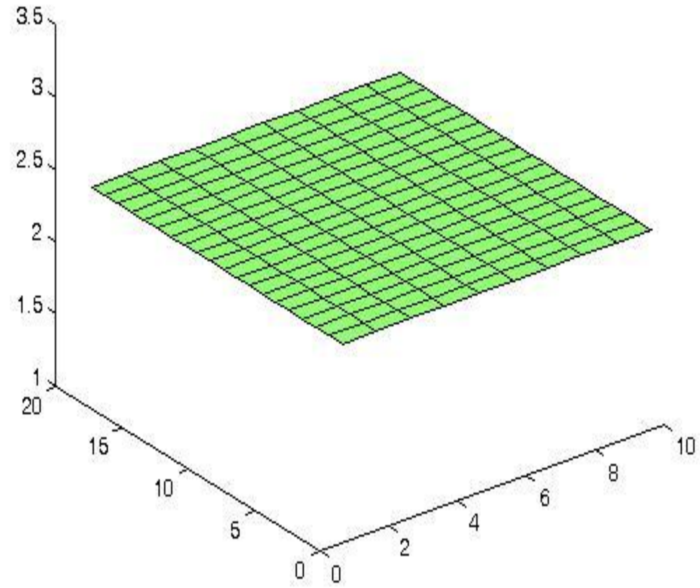
all values are 2.260

h^* average values over initial pts sweep

* N=100 players, z=1, 10 runs over

UD initial points in [0,10]^100

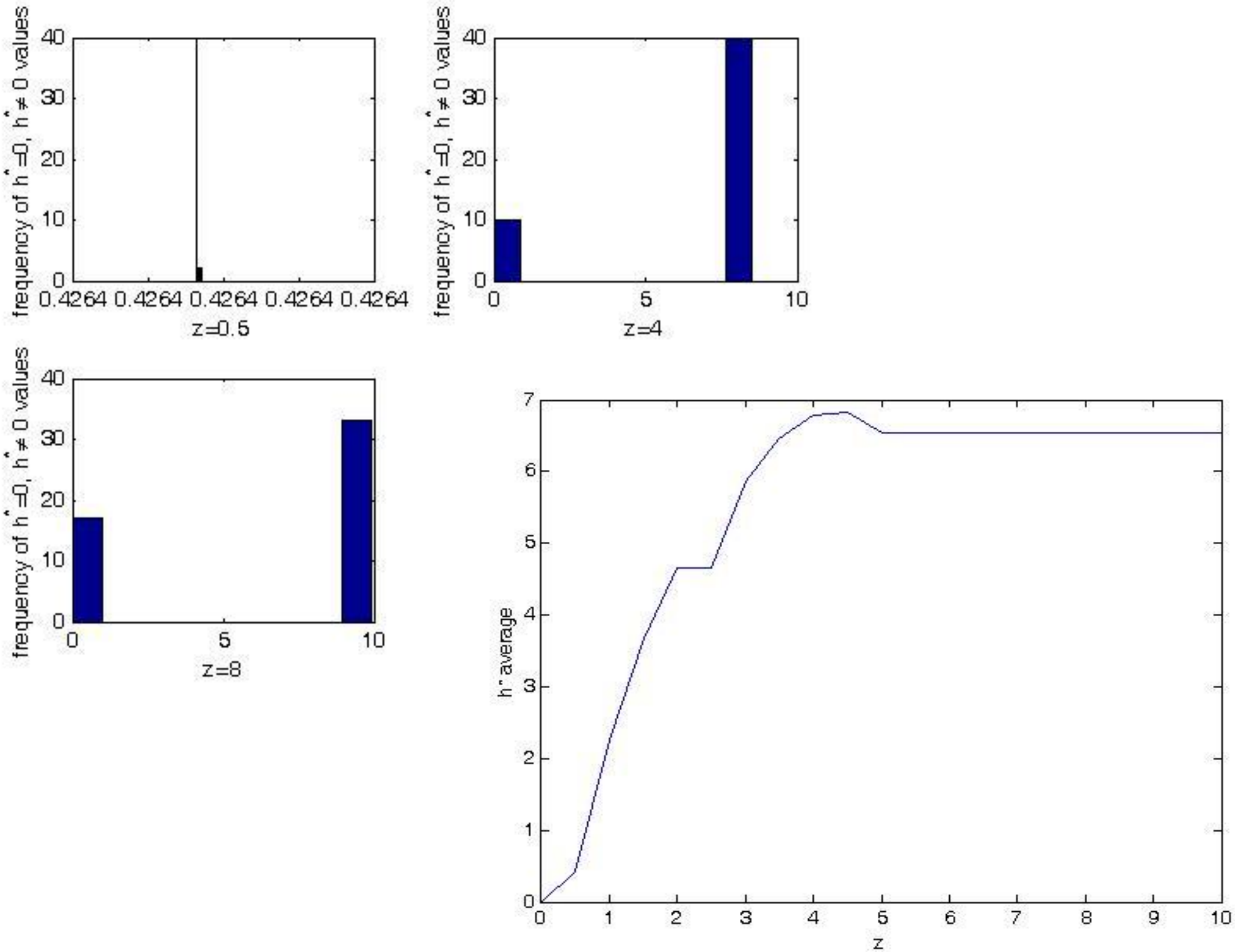




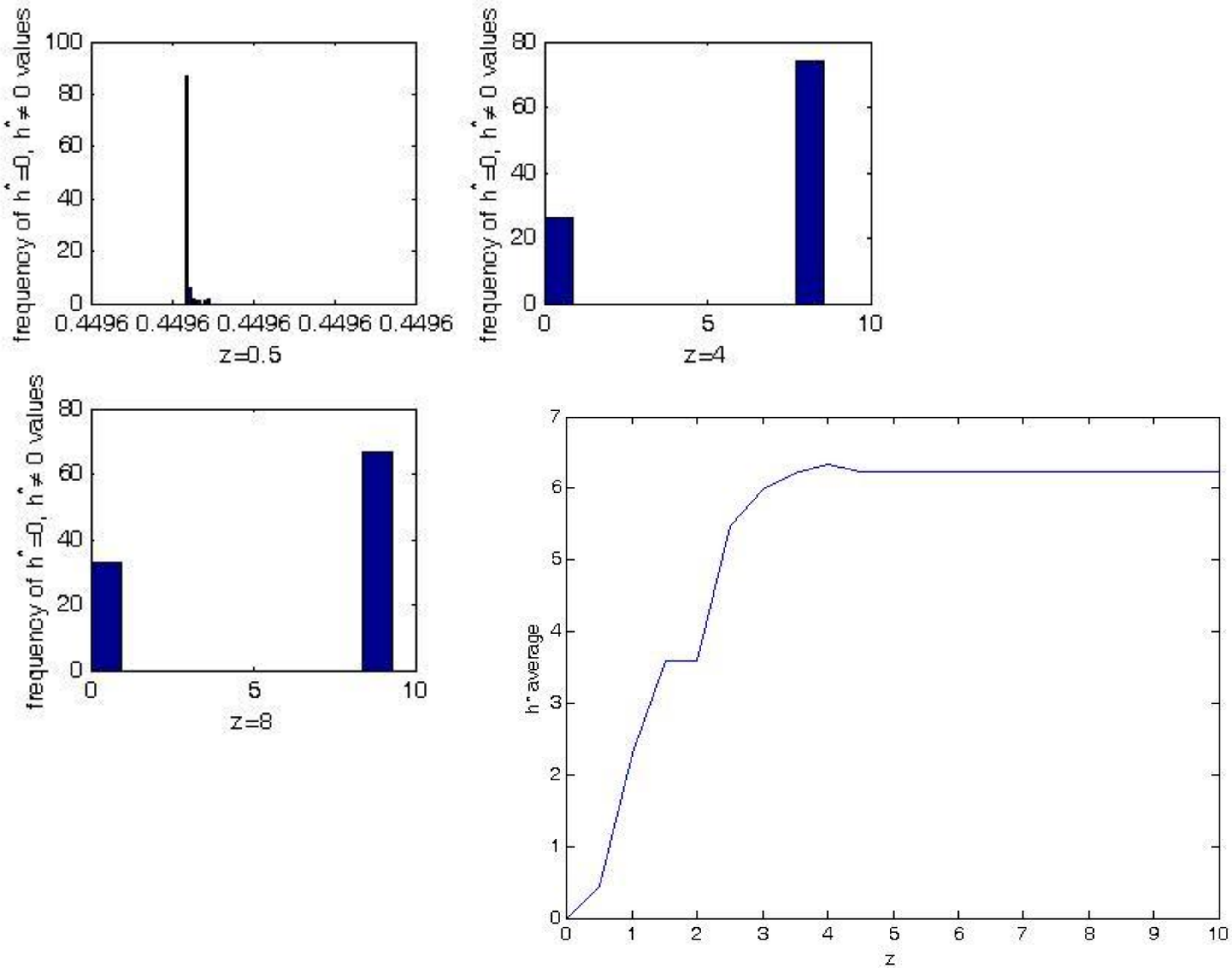
all values are 2.293

h^* average values over initial pts sweep

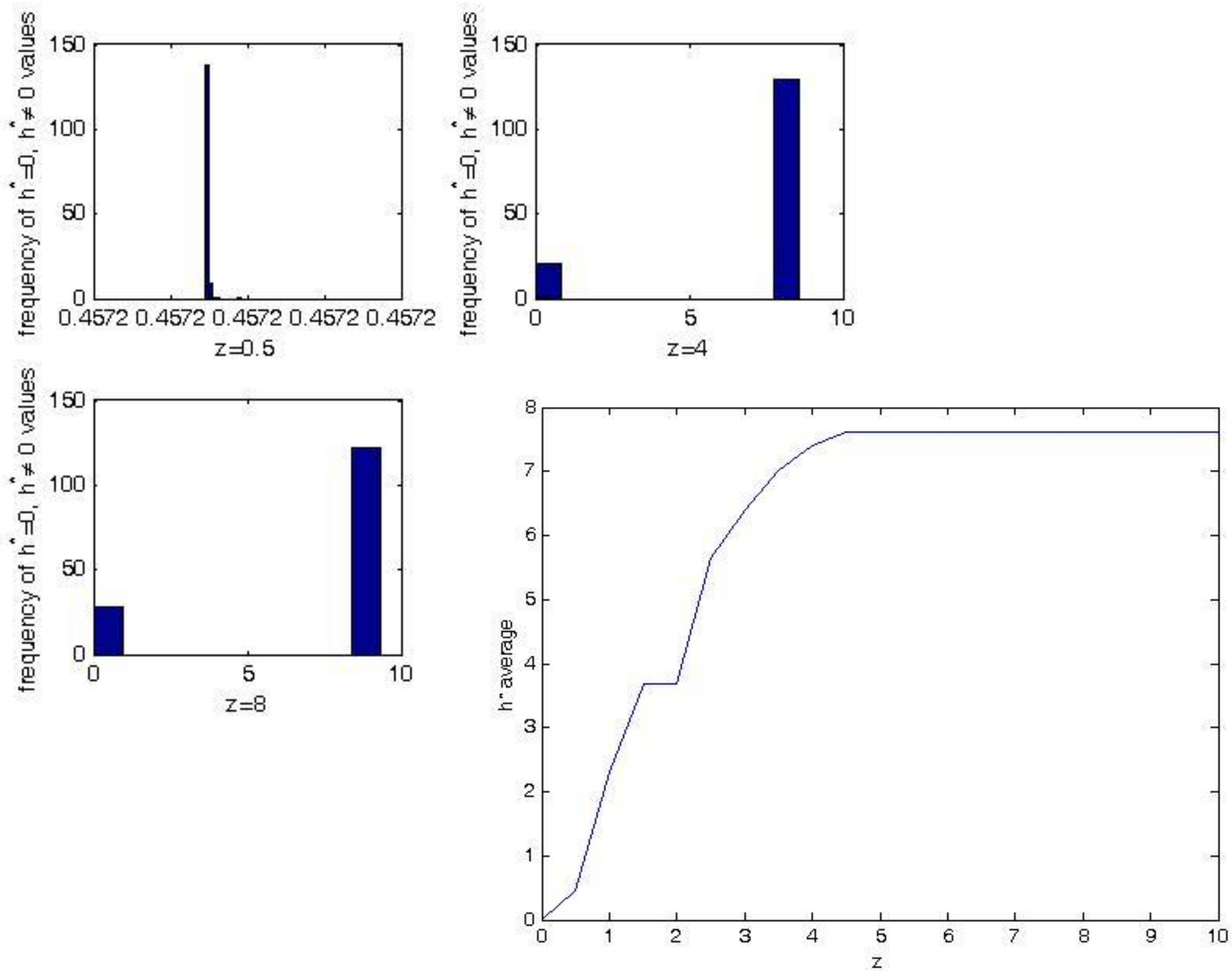
- N=50 players, z=0:0.5:10



* N=100 players, z=0:0.5:10

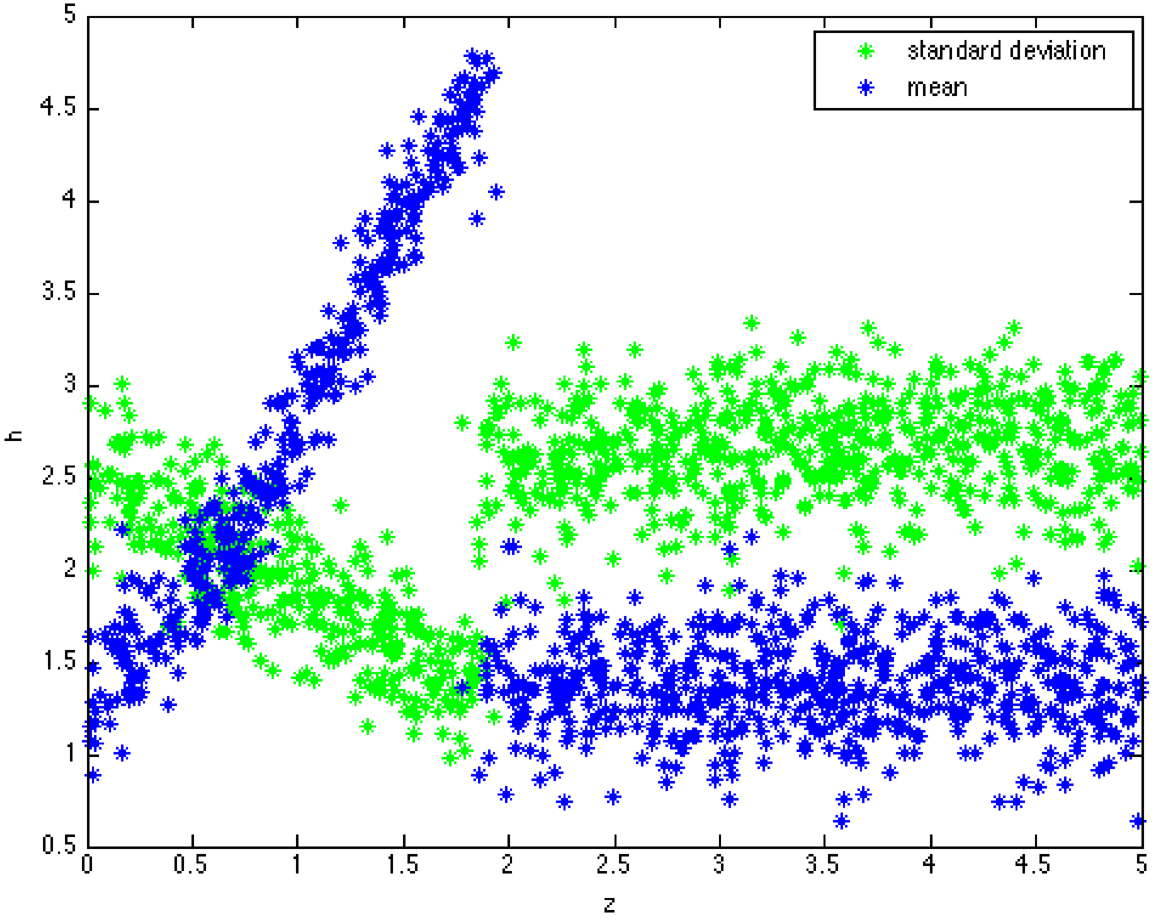


- N=150 players, z=0:0.5:10



AGENT-BASED MODEL

- We setup our population with individual values of initial help, but keeping m=10, x=0.4.
- In particular, each individual produces their own level of active help each time step. During a time step, the help received and subsequent payoff is calculated for each individual and selection occurs. For the simulations here, tournament selection is used: 10 individuals, unless otherwise noted, are randomly selected from the population and, of these, the worst two are replaced by clones of the best two.
- Mutation seems to decreases the average equilibrium help level for values of z>1, as well as it shows a different behaviour than in the game above.


standard deviation
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5

DISCUSSIONS & FUTURE WORK

- We regain some of the claims in the original paper, namely that the size of help h^* tends to increase with N, but that increase is not dramatic.
- The claim that mutants invade defectors only holds if at least two individuals start producing help at the same time. Only one is not enough to move the system away from nobody helping.
- We analyze both undermatching ($z < 1$) and overmatching coefficients ($z > 1$):

The predicted scenario of winner takes all may well happen for very large z, but there seem to be more interesting behaviour for z just slightly larger than 1:

1. There seems to be a segregation in behaviour into two types: ones that have higher and higher values of h^* $(close\ to\ m\ as\ z\ increases)$ and some who give up ($h^* = 0$).
2. It is to be investigated whether the increase in the number of players, along with variation in z, leads to a stable group of winners vs. the rest of the "give-upers".

- Last but not least: to analyze for ESS states, we need to possibly tackle the problem in a different way, as a replicator dynamics problem, or by investigating the currently described NE states for further stability properties.
- The game presented here becomes a generalized Nash game if a “resource” or “budget” constraint is present: this would translate into a shared constraint among all players, of the type :

$$h_1 + \cdots + h_N \leq \sum_{i=1}^{N} T_i$$

Where Tot = total threshold fitness in the population (an individual may help as long as the cost of helping $h_i \leq T_i$ a set threshold).

REFERENCES

- Baum, W.M., 1979. Matching, undermatching, and overmatching in studies of choice. Journal of the Experimental Analysis of Behavior 32, 269–281.
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- Williams, B.A., 1988. Reinforcement, choice, and response strength. In: Atkinson, R.C. (Ed.), Steven's Handbook of Experimental Psychology Vol. 2: Learning and Cognition 2nd ed. John Wiley & Sons, New York, NY, pp. 167–244.